THE VOLATILITY SMILE OF CANADIAN INDEX OPTIONS: ESTIMATIONS AND TESTS

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ABSTRACT

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This paper investigates the volatility smile in the Canadian Context with the TSE-60 index and the options on this underlying asset for the period between November 5, 1999 and February 14, 2001. Evidence of the existence of a Canadian volatility smile is provided. A possible reason for the existence of the smile could be attributed to the non-constant variance of the underlying asset, which violates one of the assumptions of the Black-Scholes model. Consequently, it would be incorrect to assume that the risk neutral probability distribution of the underlying asset is lognormal.

The observation of a volatility smile justifies the use of different models to recover the risk neutral probability distribution of the underlying asset. In fact, a probability distribution different than the lognormal, should be used to calculate the expected discounted option payoffs at maturity in order to get the value of the options. In this paper, the distributions are recovered from the option prices with two models that are based on a lattice approach, namely the Jackwerth-Rubinstein (1996) and the Masson-Perrakis (2000) models. Then, the derived distributions are used to price options out-of-sample. The pricing performance of the two lattice models are compared to the Black-Sholes benchmark model. Overall, the Black-Scholes appears to outperform the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models.

The absence of arbitrage opportunities in the sample is also verified. Only a few arbitrage opportunities, which are small in magnitude, are found in the series of options under investigation.

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I. INTRODUCTION

The implied volatility given by the Black-Scholes (1973) model is assumed to be constant. However, empirical studies on the S&P 500 index options have shown that the implied volatility changes with the ratio of the strike price to the index price. When the implied volatility is plotted against the moneyness (ratio of the underlying asset spot price to the strike price) of the option, an upward sloping curve is observed. This is known as the volatility smile. The shape of the smile is also known to vary with the time-to-maturity of the options. A potential reason for the existence of this smile is that it may not be right to assume that the volatility of the underlying asset is constant. Consequently, the risk neutral probability distribution of the return of the underlying asset would not be lognormal, as assumed by the Black-Scholes (1973) equation. Then, a different probability distribution should be used to calculate the expected discounted option payoffs at maturity in order to get the value of European options. Some authors have investigated this explanation for the smile and have proposed different models to recover the risk neutral probability distribution from observed data based on a lattice approach.

Pricing options with the distributions derived from option prices has the advantage of including all the information available from the volatility smile. Thus, the probabilities that investors attribute to extreme events are considered. In contrast, the Black-Scholes (1973) approach only captures the information available from one option; the one used to calculate the implied volatility.

To the best of my knowledge, the volatility smile has not been studied in the Canadian context. The existence of a Canadian volatility smile remains an empirical question. If the smile exists, the question of whether the risk neutral probability distributions, recovered with the lattice approaches, improve pricing performance becomes an important as well.

In this paper, I investigate the volatility smile derived from options traded on the Canadian TSE 60 Index. These options are European. My first objective is to provide evidence of a Canadian volatility smile. To recover the risk neutral probability distributions from the series of options on the TSE 60 Index, the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models are used. The distributions are recovered from options with the same expiration date, but different strike prices. The recovered distributions are used to price options with different moneyness and time-to-maturity. The options values are also found with the Black-Scholes model and compared with the results of the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models.

In the next section, a review of the literature on the volatility smile is presented. The following section explains the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models and presents measures used to assess the pricing ability of the models. A description of the data is presented in section three and the fourth section includes the results. Section five concludes the paper.

II. LITERATURE REVIEW

The presence of the smile is documented in many empirical studies. This section presents a review of the literature on this topic and on different methods are used by academics to capture the information contained in the smile or to resolve the inconsistency of the Black-Scholes (1973) model.

Rubinstein (1994) refers to a paper by Longstaff (1990) that presents an approach to recover the risk neutral probability distribution from option prices. The method of Longstaff (1990) requires solving a system of linear equations where it is assumed that the probabilities attached to the underlying asset between two strike prices are constant. This method also assumes a finite upper bound on the distribution.

In 1994, Rubinstein presents a non-parametric approach. This is based on a lattice framework and relies on a prespecified (prior) ending nodal risk neutral probability distribution (for instance, the lognormal distribution). His approach requires the computation of a binomial tree of (n-1) steps to calculate the possible terminal payoffs of a series of options with the same time-to-maturity, but different strike prices. The current value of the underlying asset at the beginning (root) of the tree (S_o^δ) is the observed underlying price (S_o) deflated by the dividend yield (g). Hence, $S_o^\delta = S_o e^{-gt}$ where t is the time-to-maturity in years. Moreover, S^*_j (j = 1, ..., n) denotes the possible terminal underlying asset prices at maturity and r denotes the annual riskless rate. The up move factor, u, and down move factor, d, of the tree are calculated with the implied volatility σ and the time interval of each step of the tree, $\Delta t = t/(n-1)$, is as follows: $u = e^{\sigma\Delta t}$ and d = 1/u.

Rubinstein (1994) derives the risk neutral probability distribution by solving an optimization problem based on the cross-section of the option prices. This is possible because the options have a common risk neutral probability distribution. In fact, it is the distribution of the underlying asset that is the same for all options. In this optimization problem C_{ib} and C_{ia} denote the observed bid and ask prices of the option with an exercise price X_i for i = 1, ..., m. Also, P_j anf $P_j^{\prime}$ for j = 1, ..., n denote, respectively, the implied and prior risk neutral probabilities that the underlying asset reaches the ending node j of the binomial tree. The objective function of Rubinstein's optimization problem seeks to minimize the distance, in the least squares sense, between the prespecified distribution (the lognormal in his article) and the implied one:

$$Min\left\{\sum_{j}^{n}(P_j - P_j^{\prime})^2\right\} \quad 1)$$

Subject to the following constraints:

1) The probabilities cannot be negative: $P_j \geq 0 \quad (j=1,\ldots,n) \quad 2)$

2) The sum of the probabilities must be equal to one:

$$\sum_{j=1}^{n} P_j = 1 \quad (j=1,\ldots,n) \quad 3)$$

3) The risk neutrality condition requires that the sum of the possible terminal underlying asset values, weighted by their respective risk neutral probability, be equal to the future value of the underlying spot price:

$$\sum_{j=1}^{n} P_j S_j^{*} = S_0^{*} e^{rt} \quad 4)$$

4) The implied distribution must price each option between its bid and ask prices:

$$C_{ia} \geq e^{-rt} \sum_{j}^{n} P_j \, c_{ij} \geq C_{ib} \qquad (i=1,\ldots,m) \quad 5)$$

where

P_j = Risk neutral probability of the implied distribution that the underlying asset reaches the ending node j of the binomial tree
P_j' = Risk neutral probability of the prior distribution that the underlying asset reaches the ending node j of the binomial tree
S^*_j= Assumed terminal index values at option expiration induced from a (n-1) period binomial tree
t = Time-to-maturity of an option (in years)
r = Riskless rate
X_i = Exercise Price of option i
S_o^*= The current underlying price excluding dividends
c_{ij} = Max $\{0, S_j^* - X_i\}$
C_{ib}= Observed bid price of an option i
C_{ia}= Observed ask price of an option i

Upon recovering the implied distribution from options on the S&P 500 index, Rubinstein (1994) observes that after the 1987 crash, the recovered distributions are slightly bimodal and exhibit leftskewness and kurtosis when compared to the lognormal distribution assumed by the Black-Scholes (1973) model. He explains the bimodality of the curve by "crash-o-phobia". This refers to the idea that investors are more risk averse after the crash of 1987.

Jackwerth and Rubinstein (1996) examine empirically alternative objective functions that assume prespecified (prior) ending nodal risk-neutral probabilities such as the goodness-of-fit function, the absolute difference function and the maximum entropy function. As for the objective function proposed by Rubinstein (1994), the goal of the functions is to minimize the distance between the prior distribution and the posterior one (the implied distribution). Jackwerth and Rubinstein (1996) mention that the implied distributions recovered empirically with such objective functions appear to be relatively independent of the shape of the prespecified (prior) distributions. According to Rubinstein (1994) this independence is more likely to happen if the set of options is dense. However, Jackwerth and Rubinstein (1996) note that the implied distributions do not always exhibit enough smoothness to be credible. For this reason, they propose an approach that selects the implied distribution with the maximum smoothness without having to specify a prior distribution. The optimization problem, explained in detail in section III, incorporates a penalty (α) when constraints are violated. The empirical results of Jackwerth and Rubinstein (1996) provide new evidence that the shape of the implied probability distribution has changed through time. The change in shape is apparent between precrash and postcrash distributions. Consistent with previous findings on the smile, the precrash distributions are similar to the lognormal whereas the postcrash distributions exhibit leptokurtosis (more peaked) and leftskewness. Analyzing the patterns of cumulative probability, the authors find that the precrash period cumulative probability follows the lognormal distribution. However, "In the postcrash period, a decline in the index level [S&P 500] by 3(4) or more standard deviations increased to being 10 (100) times more likely under the implied than the lognormal distribution" (Jackwerth and Rubinstein, 1996, p.1630).

Masson and Perrakis (2000) take into account the market incompleteness to estimate the implied distribution. When the market is incomplete, there are more states of nature than securities. "If states of nature outnumber assets, then it will not be possible to solve the system of equations for all distributions of wealth—some patterns of wealth cannot be constructed using the existing set of assets" (Varian, 1987.p. 57). In this situation, it is impossible to price an asset by examining the no arbitrage condition because a single asset can have an infinite number of prices. This holds even at the limit of continuous trading if there are stochastic jumps in the return of the asset. (Masson and Perrakis, 2000).

Thus, to be consistent with the incompleteness of the market, the method of Masson and Perrakis (2000) does not give a single risk neutral probability distribution but two boundary distributions between which infinitely many probability distributions are possible. The goal of their optimization problem is to get the tightest pair of distributions that contains the series of options prices observed. The major advantage of this method is that it is more robust to the violation of arbitrage within the option prices and to data errors. In other words, for most methods it is necessary to reject the option prices that present arbitrage violations before proceeding to the optimization problem (as in the approaches of Rubinstein (1994) and Jackwerth and Rubinstein (1996)). The Masson-Perrakis (2000) model requires no ex-post adjustment for the raw data that are in conflict with the no arbitrage relation between options prices. Another advantage of the Masson and Perrakis (2000) method is the possibility of finding risk neutral probability distributions more frequently than with many other approaches. That is, the optimization problem converges most of the time. The goal of the Masson and Perrakis (2000) optimization problem is to minimize the distance between option values computed by using the two distributions. Masson and Perrakis (2000) apply their method to options on the S&P 500 and find that the theoretical option values are within the relevant bid-ask spreads. This finding lets the authors argue that "the two distributions provide valuable contingent claims pricing ability in practice whenever a single distribution does not exist" (Masson and Perrakis, 2000, p. 29).

Jackwerth (2000) goes one step further in the analysis of the change in the shape of the risk neutral probability distribution recovered from options on the S&P 500 index for the period April 1986 to December 1995. Based on risk neutral probability distributions and objective probability distributions, Jackwerth (2000) derives empirically the risk aversion of a representative investor across wealth from the S&P 500 index and options on this index. The risk neutral probability distributions are recovered with a modified version of the Jackwerth and Rubinstein (1996) method. The objective distributions are approximated by a kernel density based on the historical return of the S&P 500 index. This continuous function is also discretized to match the spacing of the discrete risk neutral probability distribution.

The risk aversion function recovered by Jackwerth (2000) appears to have changed around the 1987 crash. Before 1987 the risk aversion function is consistent with the economic theory. Specifically, the risk aversion is positive and monotonically downwards sloping. (One exception to this observation occurs when the wealth level, the ratio of initial endowment over final wealth, increases more than 3% per month). The situation is different after the crash. The postcrash risk aversion function is negative throughout monthly wealth levels of 0.96 to 1.01. For wealth levels greater than 0.99, risk aversion increases. The postcrash risk aversion is apparently inconsistent with general economic assumptions. Such postcrash risk aversion would imply that the utility function of the representative investor is convex, which is unlikely to be the case in reality. Jackwerth (2000) investigated a few possible explanations for the shape of the postcrash utility function. The most plausible explanation is that it is not the risk aversion function of the representative investor that has changed after 1987, but the subjective probability distribution, which is inconsistent with the observed objective. The subjective probability distribution would have changed to reflect the concern of the investors, after 1987, that further crashes could occur. In this case, the objective distribution should be leftskewed and option mispricing is possible. For instance, out-of-money puts should be undervalued in reality because the market participants attribute significant probability to states of nature that provide low returns. Such a shape for the objective distribution could be explained by the incorporation in the distribution of an abnormally large probability for a crash occurring. Because the objective distribution is estimated from the historical return of the S&P 500 index, the skewness of the distribution is not empirically observable because the index is not mispriced. As a consequence, the distortion of the objective distribution since 1987 would be reflected in the shape of the risk aversion function. Jackwerth (2000) supports this explanation for the shape of the risk aversion function by showing that it is possible to benefit from option mispricing.

Malz (1997) proposes a different method for extracting the risk-neutral probability distribution from option prices to capture the information included in the volatility smile. In his paper, he deals with options on currency in order to estimate the probability distribution of the future exchange rate. Malz (1997) uses options that are traded over-the-counter (OTC) because of their good liquidity and large transaction volume. This market quotes currency option prices in terms of implied volatility, in units called vols. When a transaction occurs, the trader uses the value of the option in vols to calculate, with the Black-Scholes (1973) equation, the price of the option traded. The OTC market for currency options has the particularity of generally setting the exercise prices of the options in terms of delta (δ) not in terms of exchange rates. Recall that the delta is the rate of change of the Black-Scholes option value with respect to the underlying asset. However, as Malz (1997) states, dealers can pass from the delta notation to the price notation because the call delta declines monotonically as the exercise price rises.

Briefly, the following illustrates Malz's (1997) method that estimates the cumulative distribution function and the probability density function. Recall that in the OTC market, traders express the value of the options in terms of implied volatility (vols) and that the exercise prices of the options are generally in terms of delta. When the exercise price is expressed in terms of delta, the implied volatility is denoted by ($\hat{\sigma}_\delta$). However, it is important to note that it is possible to estimate the implied volatility as a function of the exchange rate $\hat{\sigma}_X$ from the observed implied volatility as a function of delta ($\hat{\sigma}_\delta$).

To capture the information contained in the volatility smile, Malz (1997) uses option combinations that are traded in the market. Their prices are measured in units of vols ($\hat{\sigma}_\delta$). The author proposes a functional form that approximates the shape of the smile of the implied volatility (as a function of delta ($\hat{\sigma}_\delta$)) by using the at-the-money volatility (atm_t) and two different option combinations, namely the Strangle (str_t) and the Risk Reversal (rr_t). As Malz (1997) mentions:

> "the at-the-money volatility gives the general level of implied volatility [...]. The Risk Reversal price indicates the skewness in the smile, and the Strangle price indicates the degree of curvature of the smile" (Malz, 1997, p.24)

The Strangle consists of a long position in an out-of-money call and a long position in an out-of-money put that have the same delta. The Risk Reversal (rr_t) position consists of a long position in an out-of-money call and a short position in an out-of-money put that have the same delta. The at-the-money volatility (atm_t) is calculated from the Straddle.

The resulting functional form that approximates the shape of the volatility smile (as a function of delta ($\hat{\sigma}_\delta$)) from the three prices of option combinations (in vols) is the following:

$$\hat{\sigma}_\delta = atm_t - 2rr_t(\delta - .50) + 16str_t(\delta - .50)^2 \qquad 6)$$

where:

$\hat{\sigma}_\delta$ = The implied volatility when the exercise price of the option is expressed as a function of delta δ
atm_t = At-the-money volatility estimated from the Straddle
rr_t = Price in vols of the Risk Reversal (which is the implied volatility of the Risk Reversal when the exercise prices of the options is expressed as a function of delta δ)
str_t= Price in vols of the Straddle (which is the implied volatility of the Straddle when the exercise price of the options is expressed as a function of delta δ)

δ = Exercise price of the option considered

From this functional form, it is possible to estimate the implied volatility in function of delta δ.

Malz (1997) estimates the probability density function by taking the second derivative of the Black-Scholes (1973) pricing formula with respect to the exercise price. To use the Black-Scholes (1973) formula, the implied volatility must be expressed as a function of the exchange rate instead of delta.

From the Black-Scholes (1973) equation, delta (δ) can be expressed as a function of the implied volatility $\hat{\sigma}_X$ (when the exercise price is the exchange rate). This expression is used to substitute delta (δ) the equation 6) mentioned above. This last version of equation 6) is the implied volatility function. This function can be solved numerically to estimate the implied volatility as a function of the exercise price ($\hat{\sigma}x$). However, the author proposes to introduce this implied volatility function into the Black-Scholes (1973) equation. By differentiating this generalized Black-Scholes (1973) formula twice numerically and by multiplying it by $e^{-rate*time}$, the probability density function can be derived. Malz (1997) uses the difference quotients method to differentiate the function. He also estimates the cumulative distribution function. To verify whether his method is accurate, Malz (1997) compares his interpolated volatility smile to the actual market prices which, as mentioned previously, are quoted in vols units. His approach appears to be accurate for most call options but small errors are observed for very high or low exercise prices because trading is thinner and dealers have problems establishing the fair market price of the options. Then, the probability density functions are accurate except for the extreme tails of the distribution. Although the approach deals with the particularities of the OTC currency option market, it suffers from the fact that the whole volatility smile is interpolated from only three points (options that have deltas of 25%, 50% and 75%).

Duan (1996) attempts to capture the information contained in the smile and its term structure. His approach is based on the GARCH option pricing model. Specifically, he uses the non-linear asymmetric GARCH model (N-GARCH) proposed by Engle and Ng in 1993 that accounts for the leverage effect. To estimate the values of the coefficients of the conditional return and conditional volatility, he suggests an optimization problem that minimizes the sum of the squared residuals between the fitted N-GARCH and the Black-Scholes implied volatilities. To determine if the coefficients of the N-GARCH are accurate in describing the smile and its term structure out-of-sample, the author estimates the N-GARCH implied volatility with the coefficients of the model estimated one week prior. The author finds that the coefficients of the N-GARCH option pricing model are able to describe the smile and its term structure. On the other hand, the author notes that the optimization problem takes considerable time to converge to an admissible solution. Therefore, it is unlikely that such an approach will be used by market traders.

Dumas et al. (1998) analyze the validity of deterministic volatility functions (DVF) in describing the shape of the smile by using options on the S&P 500 index. The DVF assumes that the implied volatility is a deterministic function of the exercise price and the time-to-maturity of an option. The authors test five different structural forms for the DVF previously specified according to Taylor series approximation in exercise price and time-to-maturity. The authors mention that kernel regression could also be used to estimate the volatility function.

Dumas et al. (1998) estimate the coefficients of the DVF on a weekly basis for the June 1988-December 1993 period. The coefficients of the volatility functions are "estimated by minimizing the sum of squared dollar errors between the reported option prices and their DVF model values". Their in-sample results suggest that parsimonious volatility functions outperform more complex specifications of implied volatility. The best results are obtained with the Black-Scholes (1973) constant volatility model.

Dumas et al. (1998) assess the stability of the coefficients of the DVF by examining the standard deviation and the correlation of the coefficients of the DVF estimated on a weekly basis. They also estimate the difference in the implied volatility calculated with contemporaneous coefficients and the previous week's coefficients. If the implied parameter values of the function are the same, the DVF is assumed to be stable through time, which justifies its use in estimating the underlying asset volatility process. However, the in the sample parameter estimates of the DVF model appear unstable over time. Another approach used by the authors to test the validity of the DVF through time is to examine the out-of-sample prediction performance of the DVF. Here, they find that the DVF performance is "no better than an ad hoc procedure that merely smooths Black-Scholes implied volatilities across exercise prices and times to expiration"(Dumas et al., 1998, p. 2059). If the DFV coefficients are stable through time, they can be used to implement hedging strategies. Such a hedging strategy should present an improvement over the one implemented with the Black-Scholes (1973) model when determining the appropriate hedge ratio. The authors report better results for hedging portfolios with the Black-Scholes (1973) constant volatility model compared to the other approach that uses DVF. Dumas et al. (1998) also recover the risk neutral probability distribution of the underling asset from the estimated coefficients of the DVF. Their implied distributions do not exhibit bimodality like the one obtained with Rubinstein (1994). Dumas et al. (1998) explain this difference by stating that their DVF specifications are simpler than the specifications used within the binomial lattice framework.

Pena et al. (1999) investigate the determinants of the shape of the volatility smile. For their study they use the Spanish IBEX-35 index options from January 1994 to April 1996. The authors estimate the volatility function by regressing the implied volatility on predefined structural forms for each day of the sample. The structural forms take into account the degree of moneyness of the options and/or incorporate quadratic terms or variables to capture the shape of the smile and its possible asymmetry. This is similar to the work of Dumas et al. (1998). According to the authors, the most appropriate structural form to describe the Spanish volatility smile is the one that assumes a linear relation between the volatility and the logarithm of the moneyness (defined in the paper as the ratio of the exercise price over the Futures price). Moreover, the structural form includes a quadratic term to capture the curvature of the smile. The coefficients of this structural form are used iteratively as an independent variable in the regression. By regressing possible explanatory variables on each of the coefficients of the structural form, it is possible to find which factor is significant in the determination of the shape of the smile. The explanatory variables in the regression are based mainly on three types of economic variables:

1) Variables that characterize the underlying asset: These include the uncertainty of the underlying asset (measured as the annualized standard deviation of the index for each day in the sample) and the liquidity (measured as the natural log of the number of shares traded).

2) Variables that help to predict the future performance of the stock market: These include the relative market momentum of the Spanish economy (measured as the log relative of the Treasury Bill rate and as the log of the ratio of the three month moving average of the IBEX-35 Index over its current value).

3) Variables that characterize the option market: These include the transaction costs (measured as the bid-ask spread) and the level of activity in the options market (measured as the natural log of the number of option contracts traded).

The following explanatory variables are also considered in the regression: dummy variables for the days of the week (to investigate the existence of a smile seasonality effect), moneyness, and the time to expiration of the options.

With this regression framework, Pena et al. (1999) find that the key determinants of the shape of the smile are the transaction costs, the time to expiration, the uncertainty of the return of the underlying asset and the relative market momentum. Specifically, the authors find that the degree of curvature of the smile is positively and significantly related to the transaction costs and significantly negatively related to the historical standard deviation of the return of the underlying asset. In addition, the smile exhibits more curvature for short times to maturity.

Pena et al. (1999) also perform Granger causality tests to investigate the presence of predictive power of the economic variables presented above on the coefficients of the volatility function that capture the shape of the smile. They find that there is a bi-directional causality between the transaction costs and the curvature of the smile. However, no causality is found between the curvature and the other economic variables. However, it seems that an unidirectional causality exists between the relative market momentum and the at-the-money volatility. That is, the relative market momentum seems to cause the at-the-money volatility. Furthermore they find that the volume of trades in the options market seems to cause the slope of the smile.

Aït-Sahalia and Lo (1998) present a nonparametric kernel regression approach to estimate the State-Price Density (SPD). SPD can be compared to Arrow-Debreu securities, but in a continuum of states. This model is also a close alternative to the binomial trees approach.

The nonparametric approach of Aït-Sahalia and Lo (1998) does not have a priori restrictions on the underlying asset's price dynamics or on the shape that the SPD should take. Their method is based on an approach suggested by Breeden and Litzenberg (1978) who find an explicit expression for the SPD of European options. According to Breeden and Litzenberg (1978), the SPD is the second derivative, normalized to have an integral of one, of a call option pricing formula with respect to the strike price. Thus, to recover the SPD, Aït-Sahalia and Lo (1998) first estimate a call pricing formula that is based on variables such as the index price, the dividend yield, the risk free rate, the exercise price and the time-to-maturity of an option. They use a nonparametric kernel regression to estimate the best call option pricing formula that fits a set of observed option prices. Secondly, they derive the function twice with respect to the strike price.

According to Aït-Sahalia and Lo (1998), their approach can be used to price new, complex or illiquid securities based on the no-arbitrage condition. Their method also has the advantage of capturing relevant information from option prices, such as the volatility smile for the pricing of contingent claims. The major difference between the approach of Aït-Sahalia and Lo (1998) and the binomial tree approach is that the latter fits one set of risk neutral probabilities for each cross section of options whereas the former requires many cross sections of options over time. In addition a single function is required to estimate the SPD over time with the approach of Aït-Sahalia and Lo (1998). In fact, it is the values of the inputs of the function that change over time, not the function. Aït-Sahalia and Lo (1998) also claim that it is easier to make statistical inferences with their nonparametric approach than with the binomial tree approach. They also state that the implied tree approach has an advantage over the SPD approach. In this case, the risk neutral probability distribution recovered with the binomial lattice is consistent with all option prices at each date, which may not be the case for the SPD. This depends on the value of the parameters of the estimated function at that specific point in time.

With the mean of a Monte-Carlo simulation, Aït-Sahalia and Lo (1998) provide evidence for the accuracy of the method. Succinctly, their other main findings are that their non parametric density and that of Black-Scholes one are not equal. Ttheir SPD functions are stable over the subperiods of their sample. Moreover, the nonparametric SPD recovered from option prices on the S&P 500 for the sample period (specifically, from January to December 1993) exhibit persitent negative skewness and excess kurtosis. They also find that the shape of the volatility smile changes as the time-to-maturity of the option increases, short term maturities exhibiting a steeper slope.

Aït-Sahalia and Lo (1998) also compare the in-sample and out-of-sample fits of their model with 3 other models, which include the Black-Scholes (1973) and the Jackwerth-Rubinstein (1996) models. At first, they analyze the ability of the estimated SPD to predict the SPD of the same set of options, but 1, 5, 10, 15 and 20 trading days later. The options considered in the sample have a maturity equal or close to 6 months. They find that the best fit of the SPD is obtained with the Jackwerth-Rubinstein (1996) model for short horizons. However, the model they propose performs better on average when all horizons are considered. Secondly, they analyze the ability of the models to predict future option prices. The horizons of prediction are again 1, 5, 10, 15 and 20 trading days. The Jackwerth-Rubinstein (1996) model appears to price options well for horizons lasting less than a week. However, for longer horizons, the Aït-Sahalia and Lo

(1998) model outperforms the others. This finding lets the authors argue that: "The speed of mean reversion of the implied volatility smile to its average pattern is sufficiently slow so that over short horizons, the most recent data remain the best predictor of future data".

A detailed overview of parametric models is presented in the paper of Bakshi et al. (1997). Their work is an extensive empirical study that compares the performance of alternative option pricing models. With data on the options on the S&P500 they analyze the performance of the benchmark Black-Scholes (1973) model, the stochastic interest rate (SI) model, the stochastic volatility (SV) model, the stochastic volatility and stochastic interest rate (SVSI) model and the stochastic volatility random jump (SVJ) model. A comparison is done on the basis of the misspecification of the models, the out of sample pricing ability and the hedging error produced by the models.

Bakshi et al. (1997) use two approaches to assess the model misspecification. First, they compare the implied volatility pattern of each model across both moneyness and maturity. Secondly, they analyze if the implied parameters (return, implied volatility and interest rate) of each model are consistent with time series data. To investigate the out-of-sample pricing performance, the authors estimate the required parameters of each model from the previous day's option prices and compute the current day option prices with those parameters. They also regress the pricing error on possible explanatory variables such as the moneyness of the option, the bid-ask spread, the time-to-maturity, the volatility and the interest rate. Two types of hedging are used to assess the dynamic hedging performance of the option pricing models. The first uses the underlying asset to hedge the position. The second is a delta-neutral hedging strategy that allows for the inclusion of a discount bond and a second option contract to hedge all possible sources of risk, namely price risk, volatility risk and interest rate risk. Overall, Bakshi et al. (1997) do not find that one model consistently outperforms the others in every aspect studied (misspecification, out of sample pricing and hedging). However, they find that including stochastic volatility and random jumps improves the performance of the option pricing model.

Das and Sundaram (1999) also study the extent to which option models that allow for jumps in the return process or stochastic volatility can resolve the documented anomalies of the Black-Scholes (1973) model. Specifically, they study if each of the models is able to match the observed levels of skewness and kurtosis of the return distribution when the maturities of the option change. Their article offers a formal analysis of the skewness and kurtosis of the underlying asset. The skewness of the return of the underlying asset is associated with the correlation between the Brownian motion of the underlying asset and the volatility. Also, the kurtosis is affected by the variation of the volatility of the underlying asset. Consistent with the findings of Bakshi et al. (1997), Das and Sundaram (1999) are unable to find evidence for the idea that stochastic volatility and random jump option models are able to explain the term structure of the Black-Scholes (1973) anomalies. However, the stochastic volatility volatility model appears to perform better than the model that includes jumps into the return process.

III. METHODOLOGY

A. Canadian volatility smile

The first objective of this paper is to determine if a Canadian volatility smile exists. Thus, for many days of the sample, the Black and Scholes (1973) equation has to be inverted to estimate the implied volatility of options that have different strike prices. Specifically, in this paper, the implied volatility is estimated for all call options on the TSE-60 index in the November 5, 1999 to April 28, 2000 period. In total, it gives 8301 implied volatilities. This should be sufficient to examine the possibility of the existence of a Canadian smile. The implied volatilities are classified on the basis of the moneyness (ratio of the underlying spot price to strike price) and days to maturity of their respective options. With this classification, it is possible to examine if the average implied volatility changes for different values of moneyness. Then, comparisons of the smile for different times to maturity are feasible. This classification is the same as the one used by Bakshi et al. (1997) for their investigation of the U.S. volatility smile. The tables of implied volatilities for the TSE 60 Index and for the S&P 500 Index are presented below. The values of implied volatilities for the S&P 500 index are those of Bakshi et al. (1997) and are presented here for comparison purposes even if the dates used to compute the smile differ. The overlapping of the dates is of little importance because the comparison of the results is done on the basis of the shape of the smile, not on the individual values of the implied volatilities of the table.

Table I

TSE 60 Index
Implied volatilities from the Black-Scholes model of individual calls for the period of November 5, 1999 to April 28, 2000

	Call Options		
	Days to expiration		
S/X	<60	60-180	>180
<0.94	28.22 (550)*	22.87 (236)	21.50 (96)
0.94-0.97	22.52 (296)	21.85 (310)	21.67 (129)
0.97-1.00	22.42 (334)	22.57 (330)	22.39 (137)
1.00-1.03	22.83 (550)	23.32 (550)	23.27 (550)
1.03-1.06	24.03 (343)	24.18 (344)	23.86 (125)
>1.06	31.51 (1215)	28.96 (1740)	27.32 (466)

* The number of observations is within brackets

It can be seen in Table I that when the time-to-maturity of the options is less than 60 days, the average implied volatility decreases when the moneyness (S/X) increases up to the point when the options are at-the-money. When the moneyness is higher than 1.00, the volatility increases with increases in moneyness. However, it should be noted that this U-shape, observed for the aggregate results for maturities less than 60 days, is not a good representation of the shape of the smile of the individual series of calls. In fact, by analyzing the data more closely, it is possible to see that the shape of individual smiles is almost always increasing from low moneyness to high moneyness (upward sloping). The U-shape observed in the aggregate results for maturities less than 60 days could be explained by the fact that most of the observations for the category of moneyness less than 0.94 and time-to-maturity less than 60 days come from the month of November

1999. Then the average volatility for this category is largely affected by the value of the implied volatility of the call options for this month. Specifically, for this category, 361 of the 550 observations come from the month of November when options are deeply out-of-the-money have a relatively high implied volatility compared to the rest of the observations in this category. Fortunately, this is the only category of the table that suffers from such a bias. In the other categories, the observations are fairly well spread across the 6 months of the sample. When the time-to-maturity of the options exceeds 60 days, the average volatility increases with the moneyness (S/X), however, an exception arises when the moneyness is lower than 0.94 and when the time-to-maturity is between 60 and 180 days.) The volatility also appears to increase as the time-to-maturity decreases for deep in-the-money and deep out-of-money options.

A similar shape for the implied volatilities is observed by Bakshi et al. (1997) for options on the S&P 500 for the period June 1990 to May 1991 (see Table II). A U-shape is found for maturities of less than 60 days. When the number of days to expiration is more than 60 days, the implied volatility increases with increases in moneyness. Again, an exception arises when the moneyness is lower than 0.94 and for time-to-maturity between 60 and 180 days. Similarly, the implied volatility appears to increase as the number of days to maturity decreases. This pattern is also observed for the options on the TSE 60 index.

Table II

S&P 500 Index
Implied volatilities from the Black-Scholes model of individual calls for the period of June, 1990 to May, 1991

	Call Options		
	Days to expiration		
S/X	<60	60-180	>180
<0.94	19.70	18.81	17.55
0.94-0.97	18.23	18.24	17.70
0.97-1.00	18.65	19.25	18.37
1.00-1.03	20.57	20.64	19.55
1.03-1.06	23.37	22.02	20.58
>1.06	30.34	24.94	23.24

*(Source: Adapted from Bakshi et al., 1997, p. 2015)

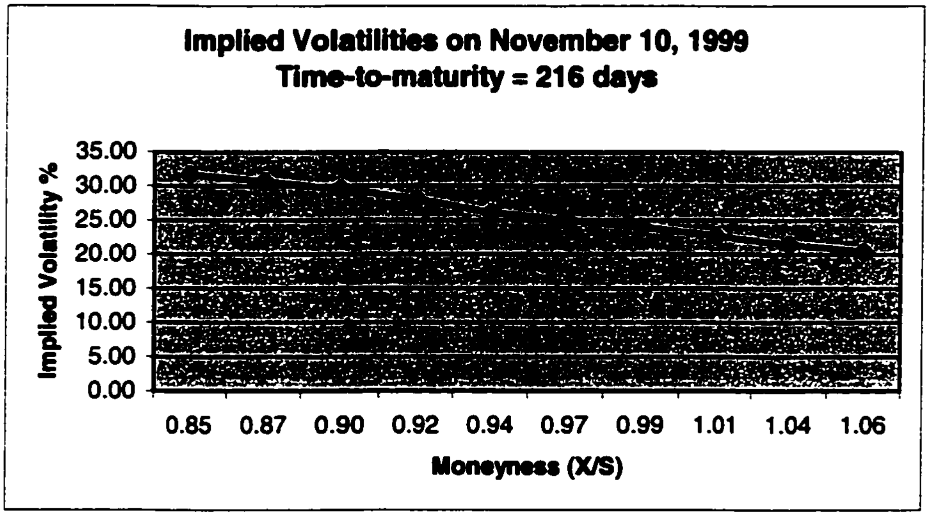
By plotting the implied volatilities against the moneyness (exceptionally defined as X/S), it is possible to see the shape of the volatility smile for each option series. Figure I presents graphs of typical smiles observed for the sample of calls of different times-to-maturity. The shape of the curve for most series is downward sloping. Again, this indicates the existence of a volatility smile in the Canadian market.[1] A possible explanation for the presence of a smile is based on the violation of one of the basic assumptions of the Black-Scholes (1973) equation, namely the constant volatility assumption (Rubinstein 1994). Therefore, it would be wrong to assume that the risk

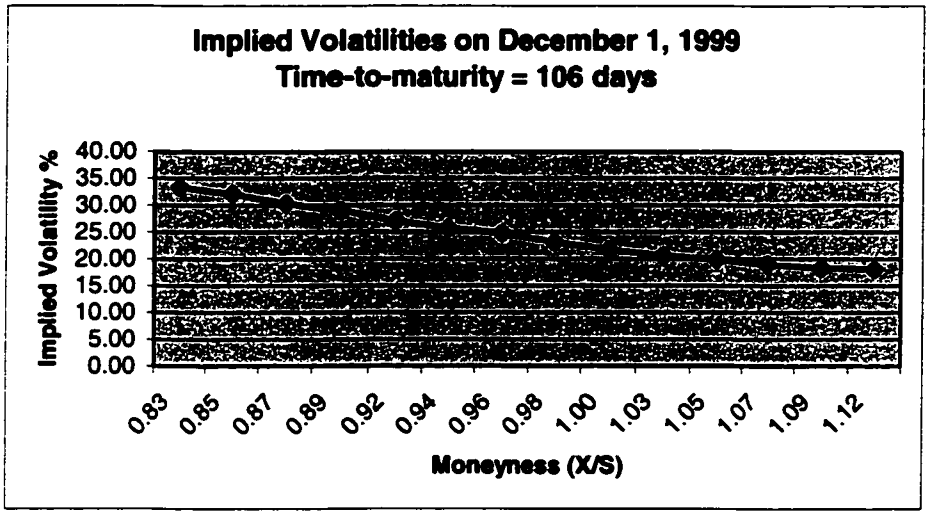
[1] The smile is not specifically investigated with put options. However, if the implied volatilities were estimated from puts, a similar shape for the smile would be obtained because of the put-call parity relationship (Bakshi et al. 1997).

neutral probability distribution of the underlying asset is lognormal. Then, observation of a volatility smile indicates the need to recover the risk neutral probability distribution of the underlying asset.

Figure I

Typical Volatility Smiles observed in the sample





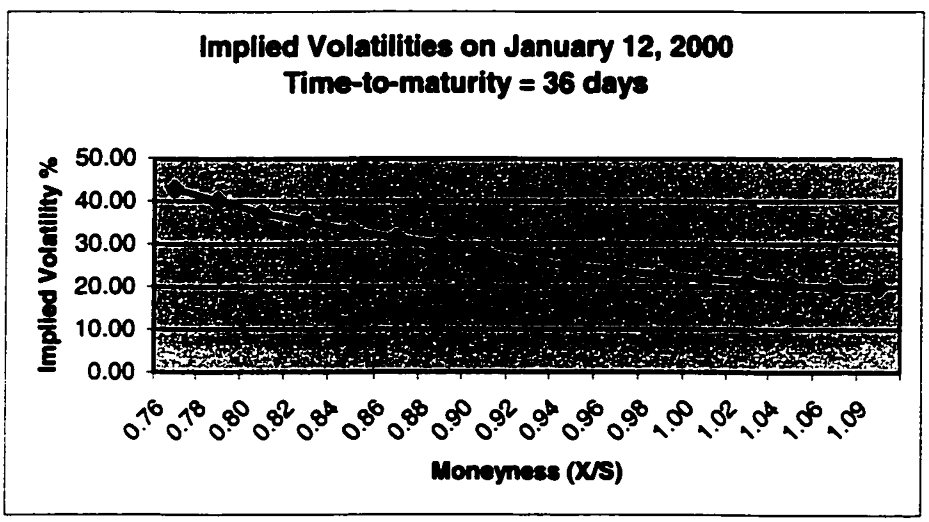
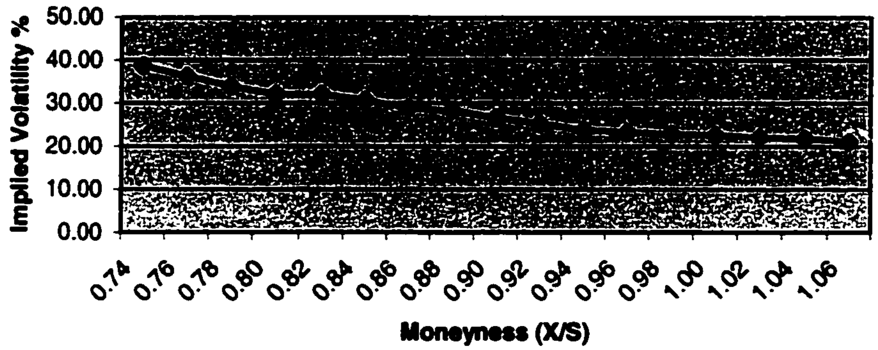
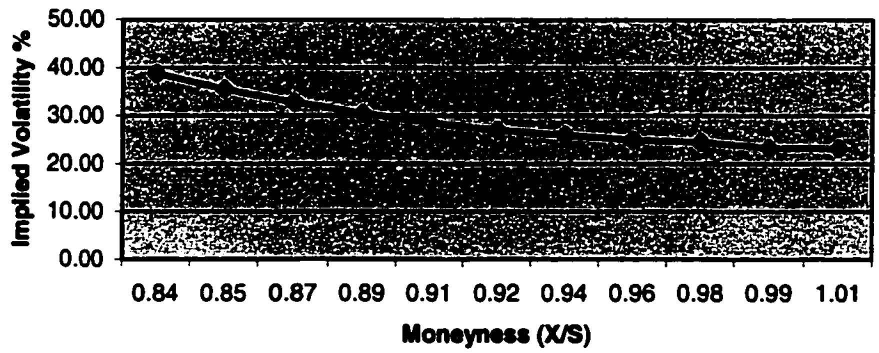


Figure I (Cont.)
Typical Volatility Smiles observed in the sample

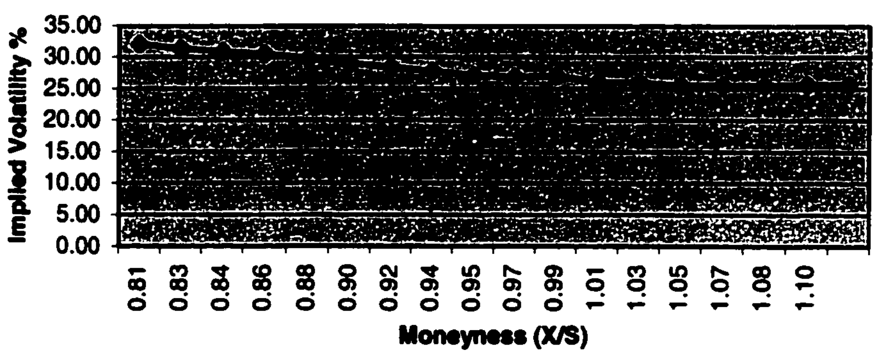
Implied Volatilities on February 1, 2000
Time-to-maturity = 46 days



Implied Volatilities on March 21, 2000
Time-to-maturity = 30 days



Implied Volatilities on April 4, 2000
Time-to-maturity = 72 days



B. Lattice approaches

Parametric methods for recovering the risk neutral probability distribution specify a priori the form of the distribution. In contrast, non-parametric methods are more flexible and capture more information on the perception of market participants because any type of probability distribution is a possible solution. The methods used in this paper to recover the risk neutral probability distribution of the underlying asset are non-parametric. These are the Jackwerth-Rubinstein (1996) and the Masson-Perrakis (2000) methods. Both are based on a lattice approach. A description of the variables used in those models follows. The notation is similar to that employed by Masson and Perrakis (2000).

In the computation of a (n-1)-step binomial tree, the current underlying asset price (S_o) has to be deflated by the dividend yield (g) because of the market participants' expectations of dividend payments throughout the life of the option. The dividend yield is assumed to be constant and known by investors. Thus, if S_o^* denotes the current asset price that excludes dividends, S^*_j (j = 1, ..., n) denotes the possible terminal asset prices (dividends excluded) at the maturity date of the option. Let t be the time-to-maturity of the option in years. Then, S_o^* is calculated as follows: $S_o^* = S_0 e^{-gt}$. The other parameters of the binomial tree are the time interval between two successive levels of the tree $\Delta t = t/(n-1)$; the implied volatility, σ; the up move factor, $u = e^{\sigma\Delta t}$; and the down move factor, $d = 1/u$. The annual riskless interest rate is denoted by r.

Observed option prices are required for the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models to recover the risk neutral probability distribution(s). The objective of both methods is to estimate the probabilities attached to each of the n possible terminal underlying asset prices of the (n-1)-step binomial tree. These probabilities must be consistent with the observed option prices. In this paper, the observed option prices consist of a series of call options that have the same maturity, but different strike prices. Let C_{ib} and C_{ia} be the bid and the ask prices of the options observed with corresponding strike prices X_i for $i = 1, \ldots, m$ and let C_i be the average of bid-ask prices of these options. I am aware of an alternative method that estimates the value of the C_i. This method is based on the average of the implied variances, the square of the implied volatility, estimated from the bid and ask prices of each option. Once the average variance is found for each option of the series, the square root of the variances is taken to find the implied volatility of each C_i. Then, the implied volatility of each option is introduced iteratively in the Black-Scholes (1973) equation to estimate the value of each C_i in the series.

For every observed option in the series, the n possible terminal option prices (payoffs) of the binomial tree are calculated. Thus, $c_{ij}= \text{Max}\ (0,S^*_j-X_i)$, ($i = 1, .., m$ and $j = 1, \ldots, n$) denote the terminal option prices at the maturity date of the option. From this cross-section of option prices, the lattice framework, and the observed option prices, the n risk neutral probability values of the binomial tree can be estimated with the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models. For these models, the risk neutral probability distribution is the same for all the options in the series because it is the distribution of the common underlying asset. With the n probabilities, it is possible to trace the shape of the empirical risk neutral probability distribution. Both approaches estimate the distribution by solving an optimization problem where the goal is to find the risk neutral probability distribution that fits the observed option prices.

1) Jackwerth and Rubinstein (1996)

As stated previously, the optimization problem in Jackwerth and Rubinstein (1996) recovers the risk neutral probability distributions from a series of option prices that expire on the same date. This optimization problem does not assume a prior distribution. Moreover, it selects the implied distribution with the maximum smoothness. Smoothness is important because the probabilities attached to similar states of nature, as described by the returns, should be close to each other. The purpose of their objective function is to:

> "find the smoothest distribution in the sense of minimizing the second derivative of P_j with respect to the underlying asset level, thereby minimizing the curvature exhibited in the implied probability distribution" (Jackwerth and Rubinstein, 1996, p.1622):

In particular, they consider:

$$\min \sum_j (P_{j-1} - 2P_j + P_{j+1})^2 \qquad 7)$$

where

P_j = Risk neutral probability that the underlying asset reaches the ending node j of the binomial tree

Jackwerth and Rubinstein (1996) state that it is more appropriate to use the midpoint of the bid-ask quotes instead of the bid and ask quotes to fit the distribution. This is to avoid dealing with an implied volatility smile that is more convex than the ones obtained with either the bid or ask prices. However, an optimization problem, based solely on the midpoint of the bid-ask quote, is less flexible than an approach allowing for the present value of the options to be in between the bid and the ask. Thus, to avoid data overfitting, the authors include constraints in the objective function. The constraint functions are added to the objective function with a penalty parameter, α, so that the value of the objective function increases each time a constraint is violated. The larger the value of α, the larger the penalty applied to the function when constraints are not respected. After experimenting with different values of the penalty parameter, the most appropriate value is determined to be $\alpha=1000$. The experimental values used in the optimization problem are α = 10, 100, 1000, 10000 and 100000. I have selected the highest value of α that allows the problem to converge. Jackwerth and Rubinstein (1996) also state that the typical value for the penalty parameter is 1000 for objective quadratic function. Hence, the α of 1000 should give accurate results without overfitting the data.

The first two constraints that are considered in the optimization problem are innocuous and stipulate that 1) the probabilities should not be negative and 2) that the sum of the probabilities must be equal to one. The third constraint insures the presence of the risk neutrality condition. That is, the present value of the underlying asset computed with the P_j must be equal to the spot price of the underlying excluding dividends. The last constraint requires that the present value of each option be equal to its respective bid-ask quote midpoint. The complete version of the objective function to optimize (including the penalties) is the following:

$$\min \sum_j (P_{j-1} - 2P_j + P_{j+1})^2 + \alpha \left\{ \begin{array}{l} \sum_j \left[\max\left[0, -P_j\right]\right]^2 + \left[\sum_j P_j - 1\right]^2 + \left[\left(\left(\sum_j P_j S_j^*\right) / e^{rt}\right) - S_o^*\right]^2 \\ + \sum_i \left[\left(\left(\sum_j P_j \max\left[0, S_j^* - X_i\right]\right) / e^{rt}\right) - C_i\right]^2 \end{array} \right\} \quad 8)$$

$(P_{-1} = P_{n+1} = 0)$

where

P_j = Risk neutral probability that the underlying asset reaches the node j of the binomial tree
α = Penalty parameter
g= Annual dividend yield
t = Time-to-maturity of an option (in years)
r= Riskless rate
S^*_j= Assumed terminal index values at option expiration induced from a (n-1) period binomial tree
S_o^*= The current underlying price, excluding dividends
X_i= Exercise price of option i
C_i= Midpoint between the bid and the ask of the option with the exercise price X_i

Before proceeding with the above nonlinear optimization problem, the option prices that violate the general no-arbitrage conditions must be removed from the data set in order to find an admissible distribution. As explained in the next section, I verify the presence of quotes that do not respect the Merton Bound and offer the possibility of implementing profitable Vertical and Butterfly spreads.[2]

The "What's Best Software" (produced by Lindo Systems Inc.) is used to solve the nonlinear optimization problem. Because of the complexity of the objective function, most optimization software packages take time to converge to a local minimum. For this reason, a limited number of binomial tree steps should be considered. For this model, a 199 step binomial tree is used. This gives 200 final node probabilities to trace out the shape of the risk neutral probability distribution. I am aware that Jackwerth and Rubinstein (1996) simplify the optimization procedure by providing an almost closed form solution to the problem. This procedure requires solving a system of equations. However, this approach requires several decisions to be made with respect to the plausibility of the many candidate risk neutral distributions. For instance, the authors suggest rejecting distributions that 1) exhibit extreme multimodalities, or 2) that have a large number of negative probabilities. The authors recommand selecting distributions with appropriate "dips" spacing, etc. Such an approach is unlikely to be used by investors that need a risk neutral probability distribution to price their options because the criteria for selecting the appropriate distribution are too subjective. For this reason, the results presented in this paper are obtained by solving the complex objective function presented above.

[2] The description of these no arbitrage conditions are presented in the next section.

2) *Masson and Perrakis (2000)*

The optimization problem of Masson and Perrakis (2000), like that of Jackwerth and Rubinstein (1996), recovers the risk neutral probability distributions from a series of option prices that expire on the same date. However, their approach does not extract a single risk neutral probability distribution, but obtains two bounds between which an infinite number probability distributions can exist. This accommodates for the possibility of market incompleteness that may result from the presence of a jump component in the returns of the underlying asset (rare unexpected events impacting the value of the index).

The Masson-Perrakis (2000) objective function attempts to find the lower and upper bounding distributions that give the tightest bounds between the option values. Four alternative objective functions are suggested by Masson and Perrakis (2000). These objective functions measure the "tightness" of the bounds. The form of the objective functions depends on the purpose of the distributions needed to compute the values of options with differing levels of moneyness. Masson and Perrakis (2000) find that when there is no restriction on the moneyness of the options that are priced with the derived distributions, the following function provides the best results on average. This is the objective function that is used in the paper.

$$Min\left\{\sum_{i}^{m}\sum_{j}^{n}c_{ij}\left(P_{ju}-P_{jl}\right)\right\} \qquad (i=1,\ldots,m,) \text{ and } (j=1,\ldots,n) \qquad 9)$$

subject to the following constraints:

10) $P_{1u} \geq P_{1l}$;

11) $\sum_{j}^{n-1} P_{ju} \leq \sum_{j}^{n-1} P_{jl}$ (j=2,...,n)

12) $\sum_{k}^{j}\left(\sum_{h}^{k} P_{hu}\right)\Delta S_k^* \geq \sum_{k}^{j}\left(\sum_{h}^{k} P_{hl}\right)\Delta S_k^*$ (j=2,...,n) and $\Delta S_k^* = S_{k+1}^* - S_k^*$

The theoretical value of the options priced with the lower bounding distribution must be less than or equal to the theoretical value of the options priced with the higher bounding distribution.

> "This provision is satisfied if the expectation of any continuous, non-decreasing and convex function C(z) defined over the assumed stock return after time T [time-to-maturity of the options] is larger under $F_u^{(k)}$ [the cumulative upper bounding distribution function] than under $F_l^{(k)}$ [the cumulative lower bounding distribution function]". (Masson and Perrakis, 2000, p.37)

The stochastic dominance constraints 10) to 12) ensure the satisfaction of this provision. These constraints are based on a theorem of the stochastic dominance of the distributions, which is demonstrated in the paper of Masson and Perrakis (2000).

13) $e^{-rt}\sum_{j}^{n} P_{ju} c_{ij} \geq C_i \geq e^{-rt}\sum_{j}^{n} P_{jl} c_{ij}$ (i=1,...,m)

The midpoint between the bid and ask of each option in the series must be between the expected value of each theoretical option value.

14) $\sum_{j=1}^{n} P_{jl} * S_j^* = S_0^* e^{rt} = \sum_{j=1}^{n} P_{ju} * S_j^*$

The risk neutrality condition 14) must be respected.

15) $\sum_j^n P_{ju} = \sum_j^n P_{jl} = 1$, 16) $P_{ju} \geq 0$, 17) $P_{jl} \geq 0$, (j=1,…,n)

The innocuous relations 15) to 17) must also be satisfied.

where

P_{ju} = Upper bound's risk neutral probability that the underlying asset reaches the node j of the binomial tree
P_{jl} = Lower bound's risk neutral probability that the underlying asset reaches the node j of the binomial tree
t = Time-to-maturity of an option (in years)
S^*_j= Assumed terminal index values at option expiration induced from a (n-1) period binomial tree
r = Riskless rate
X_i = Exercise Price of option i
S_o^*= The current underlying price excluding dividends
c_{ij} = Max $\{0, S_j^* - X_i\}$
$\Delta S_k^* = S_{k=1}^* - S_k^*$
C_i= Midpoint between the bid and the ask for the option with exercise price X_i

In contrast to the Jackwerth-Rubinstein (1996) method, this method makes no prior adjustment to the raw data to account for the potential arbitrage between option prices.

The "What's Best Software" is used to solve the optimization problem. Linear optimization problems are computationally simpler to solve than nonlinear ones. In contrast to nonlinear models, linear programs of the size I am considering can be quickly solved to optimality by commercial software. Since the Masson-Perrakis (2000) optimization problem is linear, a large number of binomial tree steps can be considered.

The model considers 499 binomial tree steps. Therefore, the discrete probability distributions are traced from 500 points.

C. Out of Sample Pricing Ability

One of the aims of this paper is to assess the pricing ability of the Jackwerth-Rubinstein (1996) and the Masson-Perrakis (2000) models. Both optimization problems find the best distribution(s) that fit the observed data. The remaining question is whether the discrete distributions can be used to price options accurately in out-of-sample tests.

Because both methods give the probabilities attached to each terminal value of a binomial tree, the options can be priced. The options are equal to the sum of the present value of their possible terminal payoffs at maturity, Max(0, S^*_j-X), weighted by their respective ending branch probabilities (P_j)s. The theoretical value of the call priced with a (n-1)-step binomial tree is equal to:

$$Call = e^{-rt}\sum_{j=1}^{n} P_j * Max(0, S_j^* - X\ ) \qquad (j = 1,\ldots n) \qquad 18)$$

where

P_j = Risk neutral probability that the underlying asset reaches the node j of the binomial tree. These probabilities come from a distribution that is recovered from the past

S^*_j= S^*_j= Assumed terminal index values at option expiration induced from a (n-1)-period binomial tree

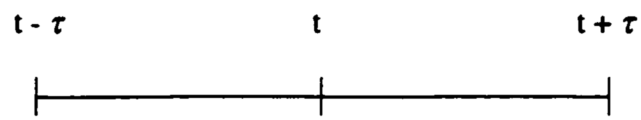
X = Exercise price of the option that needs to be price

The at-the-money implied volatility used to calculate the possible asset terminal values (S^*_j) of the options that need to be priced is the same as that used to recover the past distributions. It is important to consider the implied volatility used to recover the past distributions instead of the current at-the-money implied volatility. This approach ensures that the relative distances between the possible terminal underlying asset values of the binomial tree used to price the option are the same as the relative distances between the possible terminal underlying asset values of the binomial tree used to derive the distributions. The relative distances are the same because the up move factor, u = $e^{\sigma\Delta t}$,and the down move factor, d = 1/u, used in the computation of two binomial trees are equal if the volatility and the time interval between each step are the same.

The out-of-sample pricing ability is investigated for different time horizons. First, out-of-sample tests are performed for time horizons of 30, 60, and 90 days. Second, out-of-sample tests are be performed for a time horizon of one trading day.

1) Out-of-sample test for time horizons of 30, 60, and 90 days

It is well known in the literature that the shape of the volatility smile depends on the maturity of the options (Hull, 2000). For this reason, the first out-of-sample test uses risk neutral probability distributions that are previously recovered from a series of options that have approximately the same time-to-maturity as the options that need to be priced. In other words, if one is at time t and wants to price an option maturing at time ($t + \tau$), the following procedure is used. At time ($t - \tau$) the observed data on a series of options with different strike prices and a maturity date of time t provide the probability distribution. This distribution is then used to price the out-of-sample option, maturing at time (t + τ), at time t. For example, if the option that needs to be priced today has a time-to-maturity of 90 days, the probability distribution selected is recovered 90 days before today from a series of options that have a time-to-maturity of 90 days.



The sample of options that need to be priced includes 26 series for a total of 391 options. A series of options is defined as all the options that mature at the same date, but have different strike prices. All the strike prices available for the 26 series are considered. For these series, it is possible to find a risk neutral probability distribution that has approximately the same maturity as the option that needs to be priced. Only maturities of 30, 60, and 90 days are considered.

2) Out-of-sample test for a time horizon of one trading day

The second approach to pricing the options consists of using risk neutral probability distributions recovered one trading day before the options are priced. Because the out-of-sample time horizon is short, it is possible to recover the distribution from a series of options, which contains the option that needs to be priced. This is possible given that a series of options that includes the option that needs to be priced exists at that time. Thus, if the risk neutral probability distribution is fairly stable for a horizon of one trading day, the theoretical option values computed using the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models should be in between the bid and ask quotes.

For this out-of-sample test, the up move factor, u = $e^{\sigma\Delta t}$,and the down move factor, d = 1/u, used in the computation of the binomial tree to recover the distribution and the one used to price the option differ slightly because the time interval between two successive levels of the tree, Δt, is not exactly the same. Therefore, the relative distances between the possible terminal underlying asset values of the binomial tree used to price the option and the relative distances between the possible terminal underlying asset values of the binomial tree used to derive the distributions are slightly different.

Only risk neutral probability distributions recovered from options that have a maturity of 30, 60 and 90 days are considered for this out-of-sample pricing ability test. Practically, it means that if the distribution is recovered from a series of options with a maturity of 90 days, the options that need to be priced one day after mature in 89 days. The sample of options that need to be priced includes a total of 386 options.

3) <u>Measurements for the pricing ability</u>

The following measurements are used to assess the pricing ability of the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models:

1) The Distance Outside the Spread (DOS) is the valuation error outside the bid-ask spread of the option. If the theoretical value of the option given by the model is below (above) the option's bid (ask) price, the DOS will be equal to the difference between the theoretical value and the bid or the ask, whichever is applicable. If the theoretical value is inside the bid-ask spread, the DOS equals zero. A negative (positive) value for the DOS indicates that the option is undervalued (overvalued) by the model. Two DOS are computed for every option with the Masson-Perrakis (2000) method, that is one from each boundary distribution. The DOS is similar to the mean outside error (MOE) measure used by Dumas et al. (1998) to assess the quality of their Deterministic Volatility Function (DVF) models. In the fourth section, only the average of the absolute value of the DOS is presented. The average of the absolute value of the DOS is a better measure of the pricing ability for aggregate results because negative and positive pricing errors cannot cancel each other out in the sample. In other words, this aggregate absolute measure indicates the pricing accuracy without considering whether the model undervalues or overvalues the option price.

2) Because the value of the option varies with the value of the strike price, a relative measure of the average of the absolute value of the DOS is required. For this reason, the average absolute value of the DOS measure is presented as a percentage of the mid-point of the bid-ask price of the option.

3) The proportion of theoretical option values that are consistent with the observed option prices (between the bid and ask prices) will be calculated for the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) methods. In the same way, the proportions of undervalued or overvalued theoretical option prices is also presented. The results for this measure are only presented for the out-of-sample test that uses the distributions that are recovered from a series of options that has the same maturity as the options that need to be priced.

Because the Black-Scholes (1973) model remains one of the most popular approaches to pricing options, the options are also priced with this model for comparison purposes. The three measures presented above are also calculated for this model.

For the out-of-sample test for time horizons of 30, 60, and 90 days, the option values are computed with the Black-Scholes (1973) model with the implied volatility of the at-the-money option of the series from which the option to price comes from. This option is taken from the same series as that of the option that needs to be priced.

For the out-of-sample test that considers a one trading day horizon, the option values are computed with the Black-Scholes (1973) model using the previous day's implied volatility. Hence, the options of a series are priced with different values of implied volatilities, which depend on the values of the strike prices of the options. To a certain extent, this approach takes into account the shape of the smile to find the appropriate values of implied volatility that are necessary for pricing the options. Once the appropriate value of implied volatility is known, the Black-Scholes (1973) formula is used to price the option of interest.

IV. DATA SELECTION

A. Description of the data

The call options on the TSE 60 Index are collected from the web site of the Montreal Exchange (www.me.org). TSE 60 options are European and they expire on the third Friday of the month of maturity. Their ticker symbol is SXO. The daily closing bid and ask option prices are available on the web site after November 5,1999. The sample of call option prices considered in this paper cover the period from November 5,1999 through February 14, 2001. After February 14, 2001, the number of series of options with different maturity dates is too limited to be a part of the sample. The expirations of the options are measured as the number of calendar days between the spot price trading date and their expiration date.

The data on the riskless rate of interest are collected from Datastream. The 1 month, 2 month, 3month and 6 month maturity T-Bill rates are available. When the maturity of an option, t, is between two T-Bill maturities, the appropriate riskless rate is estimated by interpolation. In the same way, the interest rate (R) from the Spot-Futures parity relation, explained below, is interpolated when the maturity of the Futures contract, T, is between two maturities.

TSE 60 Index prices are derived from the Futures prices on the TSE 60 (F_o). This is based on the parity relation that states that the spot Index value (S_o) is equal to the present value of the Futures Prices, including dividends.

$$S_0^* = \frac{F_0}{e^{(R-g)T}} \qquad 19)$$

The settlement prices of Futures on the TSE 60 index have are collected from the web site of the Montreal Exchange and the dividend yield from the Toronto Stock Exchange. The ticker symbol for the Futures on the TSE 60 index is SXF. Several authors argue that it is more appropriate to use the Index Futures prices rather than the Index Spot prices because of the empirical evidence showing that the Spot market lags the Futures market. One possible explanation for this lead-lag relationship is that traders prefer trading in the Futures market instead of cash market because it is less costly (see Kawaller et al. 1987 and Chan 1992). However, the main reason for using the Futures instead of the Spot prices in this paper is based on the non-simultaneity of the closing times of the index and options markets. The TSE 60 Index is traded on the Toronto Stock Exchange and the closing time for this Index is 4:00 PM. The options on the TSE 60 are traded on the Montreal Exchange, and the closing time for the options is 4:15 PM. The transactions must be simultaneous for the Index and the Options because the option value directly depends on the value of the underlying asset. The Futures on the TSE 60 is traded on the Montreal Exchange and its closing time matches that of the options market. Thus, the use of Futures eliminates the issue of nonsynchronous trading.

The Futures on the index for each day is selected to ensure that the trading volume of the Futures is large enough to reflect new information in its price. The general rule is to select the Futures with the shortest maturity, but only if this maturity is longer than 1.5 months. However, for certain periods of time, only one Futures is available and thus, Futures with maturities shorter than 1.5 month are included in the sample on an exception-to-the-rule basis.

The Black-Scholes (1973) call option formula is used to find the implied volatilities required to compute the binomial tree values. The equation has to be solved numerically to estimate the implied volatility. The call option formula is the following:

$$Call = S_0^* N(d_1) - Xe^{-rt} N(d_2) \qquad 20)$$

where

$$d_1 = \frac{\ln(S_0^*/X) + (r + \sigma^2/2)t}{\sigma\sqrt{t}} \qquad 21)$$

$$d_2 = d_1 - \sigma\sqrt{t} \qquad 22)$$

where $N(d_i)$ $i=1, 2$ is the cumulative probability distribution function of a standardized, normally distributed, random variable. It provides the probability that the random variable is less than or equal to d; X is the exercise price of the option; r is the riskless rate; t is the time-to-maturity of the option; and σ is the implied volatility that needs to be estimated.

Because the spot prices of the index (derived from the Futures prices) are often in between the available strike prices, the implied volatilities of the at-the money options are estimated with interpolation for each series. The implied volatility and the variance of the bid and the ask prices of the two nearest at-the-money call options of the series are initially estimated initially. The average of the variance of the bid and ask have are calculated for the two nearest at-the-money options. I use the variance because it is recognized as an estimate of the true variance in many option models that incorporate transaction costs (see Leland 1985 and Boyle-Vorst 1992). Then, the variances are interpolated based on the distance between the strike prices and the spot asset price. The interpolation of the variance between the two options are performed for all the series of option. This procedure is particularly relevant when the slope of the volatility smile is steep. Finally, the square root of the interpolated variance is taken to find an appropriate measure of the at-the money implied volatility.

B. Arbitrage violations

To find an admissible distribution, the data used for the method of Jackwerth and Rubinstein (1996) must be cleaned to remove observations that violate the no-arbitrage conditions. These authors suggest that such data be deleted from the sample or adjusted to respect the no-arbitrage conditions. In this paper, the arbitrage relations that are verified are violations of the Merton lower bound and the presence of profitable Butterfly and Vertical Spreads. The put-call parity relation is not verified because the put prices are not required in the recovery of the risk neutral probability distribution. The no-arbitrage conditions are verified on a daily basis for the period of November 5,1999 to

February 14, 2001. All the series of options are examined. Even if the risk neutral probability distributions of specific times to maturity are estimated in this paper, a broad verification that includes every day and every series of options available is highly recommended. In fact, the presence of important profitable arbitrage opportunities can be an indication of a market that is not liquid enough to support the methodology presented in the previous section. A liquid market is essential to recover the risk neutral probability distribution with a binomial tree approach because available information on the market must be captured simultaneously by the Spot and Option markets.

The Merton lower bound stipulates that if there is no dividend on the underlying asset prior to option expiration (the spot without dividend S_o^* is used here), the value of the call ask price should be higher or equal to the Spot bid price minus the present value of the exercise price.

$$\text{Merton lower bound:} \qquad C_a \geq S^*_{obid} - Xe^{-rt} \qquad 23)$$

The convexity relation verifies the convexity of three option prices having the same maturity, but different strike prices. If the strike prices are equidistant, the option with the middle strike price should have a bid price which is lower than half of the sum of the ask prices of the two adjacent options. If this is not the case, there is an arbitrage opportunity and a Butterfly Spread would generate risk free profits.

Convexity relation (strikes equidistant): $$C_{2bid} \leq \frac{C_{1ask} + C_{3ask}}{2} \quad 24)$$

If the strikes are not equidistant, a weighted sum of the option prices, based on the distances between the strike prices must be taken into account in the convexity relation. If there are three options, namely C_1, C_2 and C_3 with strike prices X1, X2, and X3 respectively, the following convexity relation should hold, otherwise a Butterfly Spread strategy could generate risk free profits.

Convexity relation (strikes non-equidistant):

$$\left(\frac{X_2 - X_1}{X_3 - X_1} * C_{3ask}\right) + \left(\frac{X_3 - X_2}{X_3 - X_1} * C_{1ask}\right) \geq C_{2bid} \quad 25)$$

The existence of profitable Vertical spreads is also verified. In the no-arbitrage case, the difference between two adjacent options prices of a series should be less than or equal to the present value of the difference between their strike prices as follows:

No arbitrage condition for Vertical spread:

$$C_{1bid} - C_{2ask} <= e^{-rt}(X_2 - X_1) \quad 26)$$

Table III

Arbitrage Opportunities for the period November 5,1999 through February 14, 2001

Butterfly Spread			Vertical Spread		
Date when arbitrage condition not respected	Maturity of the options	Arbitrage in %	Date when arbitrage condition not respected	Maturity of the options	Arbitrage in %
4/4/00	9/15/00	0.128	1/27/00	2/18/00	0.007
4/7/00	6/16/00	0.516	4/7/00	6/16/00	0.017
4/14/00	6/16/00	0.480			
5/9/00	7/21/00	1.869			
5/9/00	9/15/00	2.158			
5/9/00	12/15/00	1.483			
5/10/00	7/21/00	2.324			
5/10/00	9/15/00	2.618			
5/10/00	12/15/00	1.734			
5/11/00	7/21/00	2.167			
5/11/00	9/15/00	2.552			
5/11/00	12/15/00	1.745			
5/12/00	7/21/00	2.074			
5/12/00	12/15/00	1.610			
5/15/00	7/21/00	0.315			
5/15/00	12/15/00	0.134			
5/16/00	12/15/00	0.113			
5/17/00	12/15/00	0.075			
5/18/00	7/21/00	0.054			
5/19/00	12/15/00	0.086			
5/23/00	12/15/00	0.102			
5/24/00	12/15/00	0.090			
5/25/00	12/15/00	0.094			
5/26/00	12/15/00	0.098			
5/29/00	12/15/00	0.049			
5/30/00	12/15/00	0.120			
6/1/00	9/15/00	2.500			
6/6/00	12/15/00	2.500			
6/14/00	12/15/00	2.500			
7/28/00	8/18/00	0.080			
11/24/00	6/15/01	0.979			
11/24/00	6/15/01	0.083			
11/24/00	6/15/01	2.537			
11/24/00	6/15/01	3.287	** Note: No data were in conflict with the Merton Bound no arbitrage condition		
11/27/00	6/15/01	0.954			
11/27/00	6/15/01	3.136			

Table III presents the arbitrage opportunities for the Vertical and Butterfly Spreads for the options on the TSE-60 index. No arbitrage opportunities are found for the Merton bound. (The arbitrage opportunities are expressed as a percentage of the midpoint of the bid and ask prices of the middle option for the butterfly spread.) For the vertical spread, they are expressed as a percentage of the average of the prices of the two options considered. Table III presents the arbitrage opportunities observed in the sample. A few arbitrage opportunities exist in the sample, but they are small in magnitude. Only two of the opportunities are above 3%. When considering the transaction costs of a Vertical or a Butterfly Spread, the potential benefits, undoubtedly, become negligible. Thus, there are virtually no arbitrage violations in the TSE-60 option market for the period analyzed. Also, I note that no-arbitrage opportunities exist for the days that are used to recover the risk neutral probability distribution.

Masson and Perrakis (2000) investigate the absence of profitable butterfly and vertical spreads on options on the S&P 500 index from 1993 through 1995. Even when transaction costs are considered, they find 27.6% of all the series of the sample present at least one opportunity for riskless arbitrage (butterfly or vertical spreads). Their results are surprising when compared to those obtained for the options on the TSE-60 index. The presence of arbitrage opportunities seems to be more frequent for U.S. options than for Canadian options. Because the options market for the S&P 500 index is more liquid than the market for the options on the TSE-60 index, fewer arbitrage opportunities should have been found for the American options market. When transaction costs are not considered, Masson and Perrakis (2000) find that the convexity relation is violated in more than 40 % of the series.

V. RESULTS AND FINDINGS

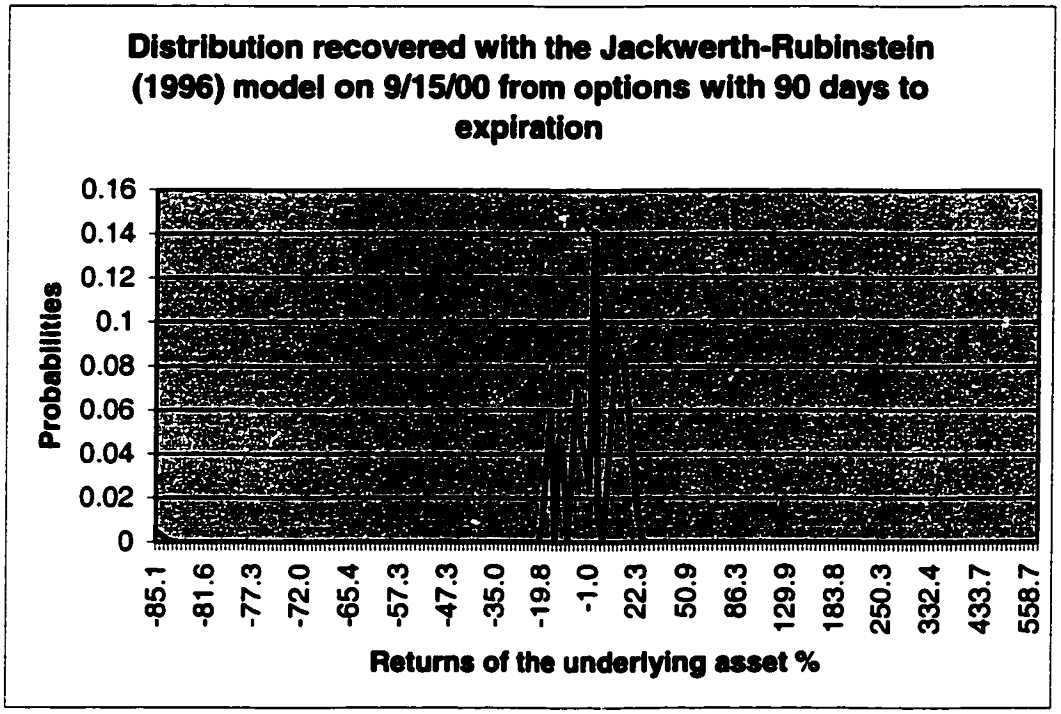
This section presents the out-of-sample pricing performances of the Jackwerth-Rubinstein (1996), Masson-Perrakis (2000) and Black-Scholes (1973) models. As mentioned previously, only options with maturities of 30, 60, or 90 days are used to test the pricing ability of the models. It has to be mentioned that with the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) methods, an admissible solution is found for every optimization problem computed. That is, all the optimization problems converge. This is partially due to the fact that no arbitrage is found in the data used to recover the risk neutral probability distributions. The convergence of the Jackwerth-Rubinstein (1996) optimization problem is also related to the value of the penalty parameter α. If the value of α is too large, it may be difficult to find an admissible solution and it increases the likelihood of overfitting the data. On the other hand, if the value of α is too small, the distributions obtained may not be accurate because the constraints inside of the objective function are easily satisfied. This highlights the importance of carefully selecting the value of the penalty parameter when implementing the Jackwerth-Rubinstein (1996) model. The value of the penalty parameter is selected with caution in this paper. Thus, the high convergence frequency of the Jackwerth-Rubinstein (1996) model cannot be attributed to choosing a poor value for the penalty parameter.

Figures II and III present typical risk neutral probability distributions obtained with the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) methods, respectively. The shape of the distribution of both models does not appear to be very smooth. Such results are surprising for the Jackwerth-Rubinstein (1996) model, which is specified to find the implied distributions with the maximum smoothness.

Tables IV and V present the option pricing errors for different categories of moneyness and times to maturity for the out-of-sample test for time horizons of 30, 60, and 90 days. Table IV contains the average of the absolute value of the DOS as a percentage of the mid-point of the bid-ask prices of the options of each category. Table V contains the average of the absolute value of the DOS (in dollars) for each category.

When the averages of the pricing errors are calculated for Table IV, a few options have an extraordinary large pricing error in percentage terms. This usually happens when the options that need to be priced have a small value because they are deeply out-of-the-money. When the DOS (in dollars) is divided by a small value, inflated pricing errors are likely to result. Hence, when extraordinary large pricing errors are observed, the specific options are considered as outliers and are not included in Table IV. For consistency, the same outliers are rejected when computing the aggregate pricing errors in table V (in dollars) and the pricing performance in table VI. A total of 34 observations are rejected out of 391 total observations. The observations considered as outliers for the out-of-sample test for time horizons of 30, 60, and 90 days are available in Appendix A. Similarly, several observations are rejected when computing the aggregate results for the out-of-sample test for a time horizon of one trading day (Table VII and VIII). A total of 5 observations are rejected out of 386 observations. These rejected observations for the second out-of-sample test are available in Appendix B.

Figure II
Typical Risk Neutral Probability Distributions



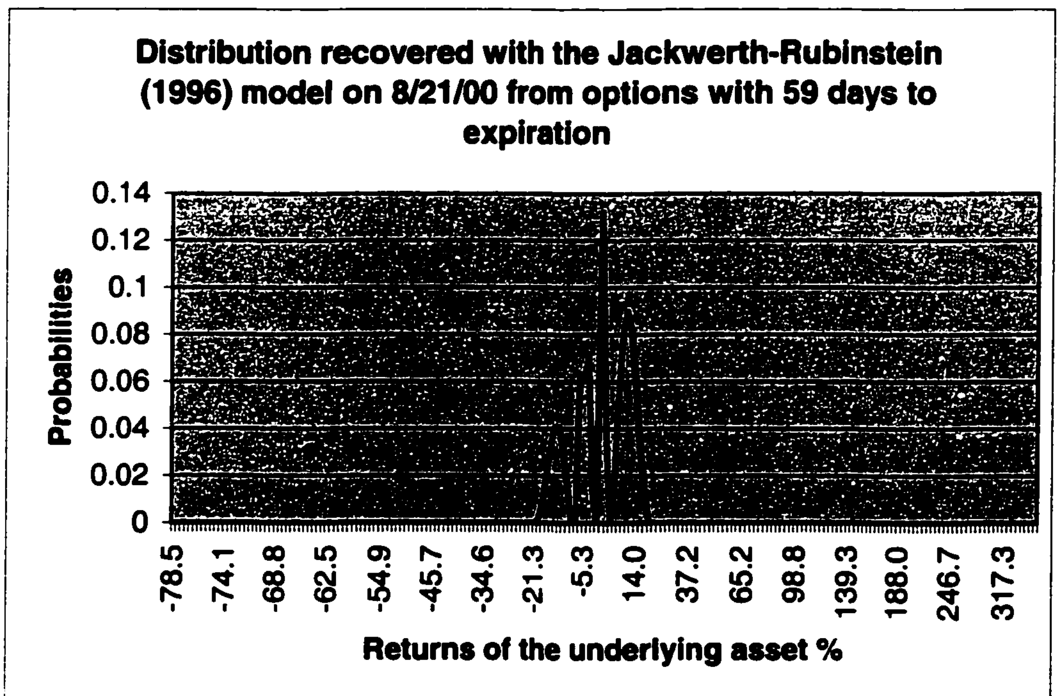
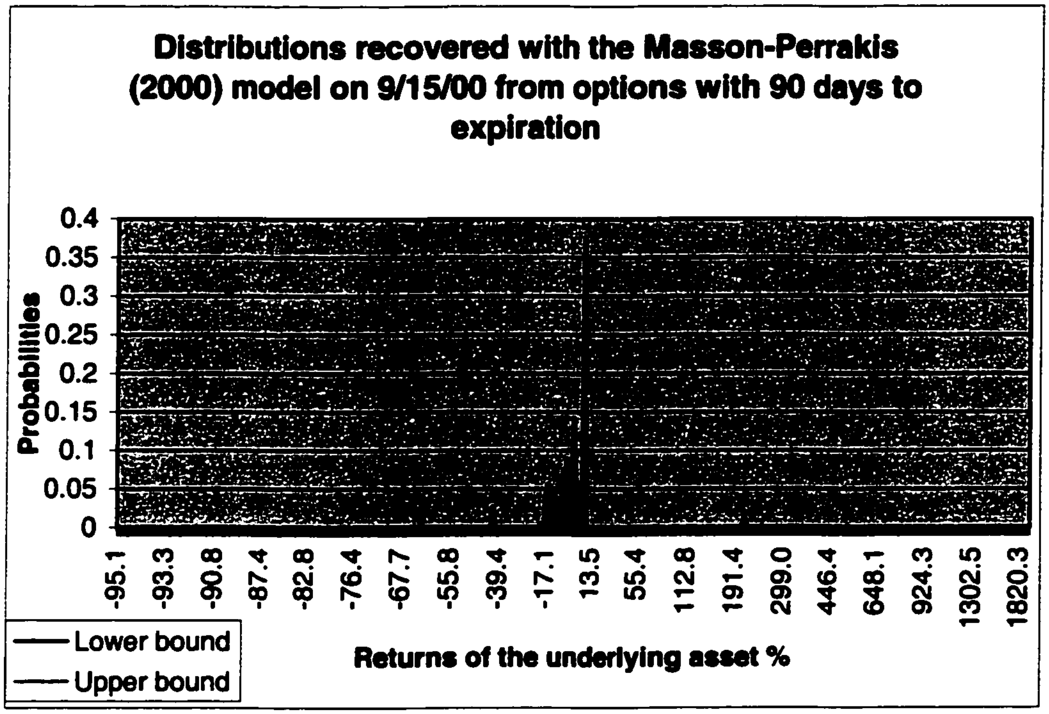
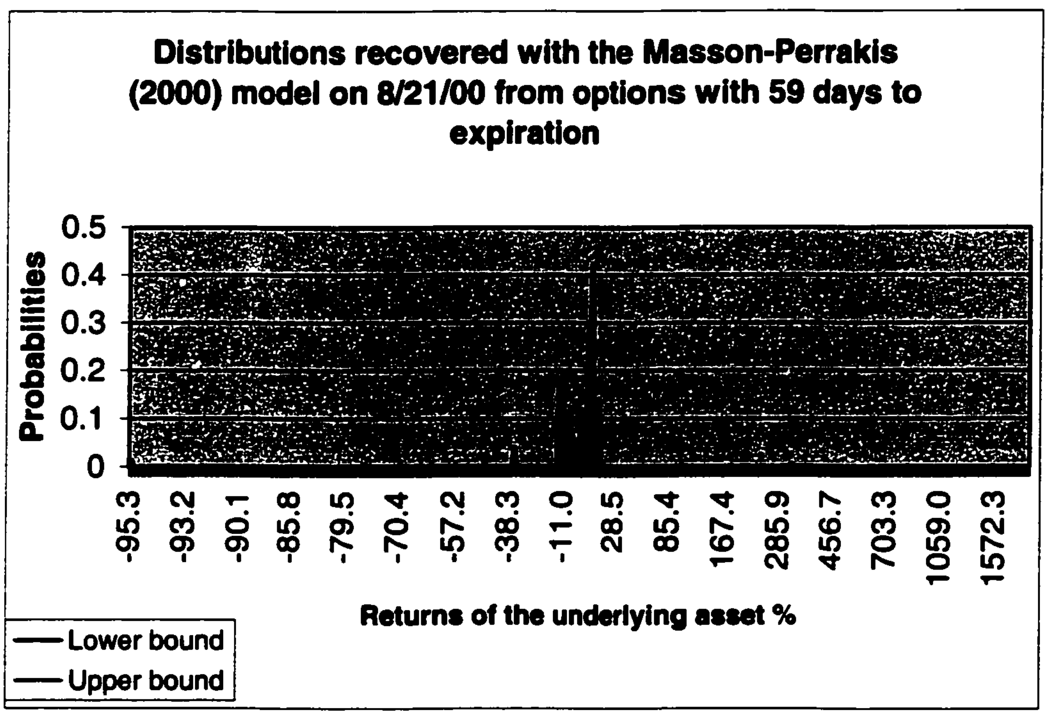


Figure III
Typical Risk Neutral Probability Distributions





The categories of Tables IV and V, which have a moneyness value of less than 1.06 and a time-to-maturity equal to 90 days, do not have many observations. Thus, the sample size does not allow for strong statistical inferences. For this reason, the results are included in the tables only for demonstration purpose.

In Tables IV to VIII, the results obtained from the Jackwerth-Rubinstein (1996) model are identified by JR. MP Lower and MP Upper identify the results obtained from the lower and upper bound distributions of the Masson-Perrakis (2000) method. BS identifies the results obtained using the Black-Scholes (1973) model.

Overall, Table IV shows that the Black-Scholes (1973) model appears to price options more accurately than the other models. The Masson-Perrakis (2000) model seems to underperform the two other models for all categories. Moreover, the pricing performance of all models generally improves as the moneyness (S/X) of the options increases. For at-the-money and in-the-money options having a time-to-maturity of 30 days, the overall pricing performance of the Black-Scholes (1973) model is slightly better than that of the Jackwerth-Rubinstein (1996) model. The only category in which the Jackwerth-Rubinstein (1996) outperforms the Black-Scholes (1973) model, both in percentage and dollar terms, is for the case when moneyness is higher than 1.06 with a time-to-maturity equal to 90 days. (Recall that no statistical inferences are made for the other categories of moneyness when the time-to-maturity is 90 days because the sample sizes are too small).

Similar results for all categories of moneyness and time-to-maturity are obtained in Table V. This table gives the amplitude of the pricing error in dollar terms. When the overall pricing errors are observed for the category "all moneyness", the Jackwerth-Rubinstein (1996) seems to perform slightly better than the Black-Scholes (1973) model. However, the average results are biased by the category of moneyness higher than 1.06 for which the Jackwerth-Rubinstein (1996) model performs better. When the categories are analyzed one by one, the Black-Scholes (1973) model clearly outperforms the two other models, except when the time-to-maturity of the options is equal to 90 days.

Table IV

Out-of-sample pricing errors for time horizons of 30, 60, and 90 days measured as an average of the absolute value of the DOS (calculated as a percentage of the midpoint of the bid-ask prices of the options)

Moneyness (S/X)	Models	Time to maturity 30 days	60 days	90 days	30,60and 90 days
<.94	JR	7.59	30.22	16.75	17.49
	MP Lower	20.63	29.85	11.39	23.30
	MP Upper	21.70	40.08	19.14	28.69
	BS	0.47	22.78	30.26	12.42
		(19)*	(15)	(4)	(38)
0.94-0.98	JR	12.48	11.62	8.72	11.63
	MP Lower	31.35	12.05	15.37	21.29
	MP Upper	32.68	15.86	19.47	24.01
	BS	11.22	5.39	25.34	10.66
		(21)	(19)	(6)	(46)
0.98-1.02	JR	4.62	8.53	8.02	6.83
	MP Lower	16.55	10.35	20.05	14.39
	MP Upper	16.90	12.63	23.95	16.11
	BS	4.63	2.64	9.88	4.56
		(22)	(23)	(8)	(53)
1.02-1.06	JR	2.77	4.13	4.06	3.48
	MP Lower	10.08	4.66	10.98	8.08
	MP Upper	11.81	5.90	14.45	9.86
	BS	1.36	2.02	13.03	3.22
		(24)	(20)	(7)	(51)
>1.06	JR	1.19	1.19	0.68	1.10
	MP Lower	2.09	3.34	5.57	3.15
	MP Upper	3.10	4.84	7.26	4.45
	BS	0.47	0.89	4.10	1.27
		(81)	(57)	(31)	(169)
All Moneyness	JR	4.02	7.62	4.16	5.39
	MP Lower	10.93	8.94	9.78	10.00
	MP Upper	12.01	11.84	12.70	12.05
	BS	2.50	4.45	10.19	4.43
		(167)	(134)	(56)	(357)

* The number of observations for each category is within brackets

Table V

Out-of-sample pricing errors for time horizons of 30, 60, and 90 days measured as an average of the absolute value of the DOS (in dollars)

Moneyness (S/X)	Models	Time to maturity: 30 days	60 days	90 days	30,60and 90 days
<.94	JR	0.31	2.35	2.87	1.39
	MP Lower	0.65	2.20	2.02	1.41
	MP Upper	0.69	3.03	3.37	1.90
	BS	0.02	1.56	5.31	1.19
		(19)*	(15)	(4)	(38)
0.94-0.98	JR	1.12	1.74	2.04	1.50
	MP Lower	2.70	2.10	3.90	2.61
	MP Upper	2.78	2.51	5.00	2.96
	BS	0.81	0.83	6.93	1.62
		(21)	(19)	(6)	(46)
0.98-1.02	JR	0.86	2.22	2.60	1.71
	MP Lower	3.04	2.97	7.11	3.62
	MP Upper	3.10	3.51	8.58	4.11
	BS	0.71	0.67	3.87	1.17
		(22)	(23)	(8)	(53)
1.02-1.06	JR	0.93	1.66	2.06	1.37
	MP Lower	3.47	1.97	5.79	3.20
	MP Upper	4.12	2.46	7.66	3.95
	BS	0.40	0.79	7.09	1.47
		(24)	(20)	(7)	(51)
>1.06	JR	0.93	0.86	0.54	0.84
	MP Lower	1.46	3.14	5.72	2.81
	MP Upper	2.31	5.21	7.43	4.23
	BS	0.30	0.70	3.80	1.08
		(81)	(57)	(31)	(169)
All Moneyness	JR	0.87	1.50	1.35	1.19
	MP Lower	2.02	2.68	5.47	2.81
	MP Upper	2.55	3.88	7.07	3.76
	BS	0.40	0.82	4.66	1.23
		(167)	(134)	(56)	(357)

* The number of observations for each category is within brackets

Table VI presents the proportion of theoretical option prices that are undervalued, overvalued or correctly estimated by the models. To estimate the frequency of accurate pricing, the theoretical values are compared with the observed bid and ask quotes of each option. Because the volatility smile exists, the Black-Scholes (1973) model should, on average, undervalue (overvalue) the in-the-money (out-of-the-money) calls. Similarly, the model should overvalue (undervalue) in-the-money (out-of-the-money) puts, but this case is not investigated. In Table VI, the Black-Scholes (1973) model clearly overvalues the out-of-the-money options. For in-the-money options, the proportion of options that are overvalued is smaller. That is, only 10% of the options are overvalued for ratios of moneyness higher than 1.06 with the Black-Scholes (1973) model. Tables IV and V show that the overvaluation is small in magnitude. Also, Black-Scholes (1973) appears to price correctly deep in-the-money options more frequently. That is, 86.4% of the deep in-the-money options are correctly priced. The Jackwerth-Rubinstein (1996) model appears to undervalue the out-of-the-money and the at-the-money options. Meanwhile, it appears to overvalue the in-the-money options. This model also appears to price correctly the deep out-of-the-money and deep in-the-money options more frequently than the options with other ratios of moneyness. The Masson-Perrakis (2000) model overvalues deep out-of-money options and in-the-money options. However, the model prices correctly most of the options that are deeply in-the-money. That is, more than 81% of the deep in-the-money options are correctly priced with Masson-Perrakis (2000). Overall, the Black-Scholes (1973) prices correctly the options 69.47% of the time. For the Jackwerth-Rubinstein (1996) method, this happens 22.69 % of the time and for Masson-Perrakis (2000), more than 57 % of the time. Then, it seems that the Masson-

Perrakis (2000) model outperforms the Jackwerth-Rubinstein (1996) model. Recall that Table VI does not consider the amplitude of the mispricing, but instead considers the frequency at which each model prices the options correctly. With this regard, Masson-Perrakis (2000) appears to outperform Jackwert-Rubinstein (1996), but when the size of pricing errors is considered (Tables IV and V), the opposite is true.

Table VI

Proportion of theoretical option prices that are undervalued, overvalued or correctly estimated by the models for the out-of-sample test for time horizons of 30, 60, and 90 days

Moneyness (S/X)	Model	Undervalued %	Overvalued %	Between bid-ask %	Number of Observations
<.94	JR	55.3	7.9	36.8	38
	MP Lower	28.9	42.1	28.9	38
	MP Upper	26.3	50.0	23.7	38
	BS	10.5	42.1	47.4	38
0.94-0.98	JR	63.0	23.9	13.0	46
	MP Lower	26.1	34.8	39.1	46
	MP Upper	21.7	41.3	37.0	46
	BS	8.7	37.0	54.3	46
0.98-1.02	JR	56.6	30.2	13.2	53
	MP Lower	22.6	32.1	45.3	53
	MP Upper	18.9	37.7	43.4	53
	BS	9.4	32.1	58.5	53
1.02-1.06	JR	43.1	51.0	5.9	51
	MP Lower	25.5	33.3	41.2	51
	MP Upper	21.6	39.2	39.2	51
	BS	9.8	35.3	54.9	51
>1.06	JR	21.9	47.9	30.2	169
	MP Lower	7.7	10.7	81.7	169
	MP Upper	6.5	12.4	81.1	169
	BS	3.0	10.7	86.4	169
All Moneyness	JR	38.9	38.4	22.7	357
	MP Lower	17.1	23.5	59.4	357
	MP Upper	14.6	27.7	57.7	357
	BS	6.4	24.1	69.5	357

I am aware of trend presented in recent option pricing literature that compares the performance of alternative option pricing models to a Black-Scholes (1973) model that is implemented using an implied volatility that is smoothed across exercise prices and days to expiration. Many market makers use this approach to price options instead of taking the at-the-money implied volatility. (see Dumas et al. (1995) and Bakshi et al. (1997)). In this case, the smoothing of the smile is done with a Best Fit approach. Such an approach finds a single implied volatility value that minimizes the sum of the differences between the Black-Scholes prices and the observed prices of a series of options that have different strike prices. Another practice consists of using an implied volatility value that comes from an implied volatility matrix. This matrix contains the implied volatility for different levels of option moneyness and time-to-maturity. With this approach, the implied volatility selected is the one that matches the moneyness and time-to-maturity of the option that needs to be priced. The results presented in Tables IV, V and VI are based on theoretical option values that are computed with the implied volatilities of at-the-money options with the same maturity as the options that need to be priced. The out-of-sample price prediction results for time horizons of 30, 60, and 90 days (Tables IV,V and VI) suggest that the Black-Scholes (1973) outperforms Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000). Thus, using more complex approaches to estimate the implied volatility from the smile brings no additional evidence.

Table VII

Out-of-sample pricing errors for a time horizon of one trading day measured as an average of the absolute value of the DOS (calculated as a percentage of the midpoint of the bid-ask prices of the options)

Moneyness (S/X)	Models	%
<.94	JR	17.44
	MP Lower	3.02
	MP Upper	2.91
	BS	1.11
		(30)*
0.94-0.98	JR	18.58
	MP Lower	2.37
	MP Upper	2.65
	BS	0.85
		(45)
0.98-1.02	JR	9.81
	MP Lower	1.79
	MP Upper	1.85
	BS	1.62
		(55)
1.02-1.06	JR	5.55
	MP Lower	1.43
	MP Upper	1.44
	BS	1.19
		(51)
>1.06	JR	2.30
	MP Lower	0.51
	MP Upper	0.52
	BS	0.48
		(200)
All Moneyness	JR	6.93
	MP Lower	1.24
	MP Upper	1.27
	BS	0.83
		(381)

* The number of observations for each category is within brackets

Table VIII

Out-of-sample pricing errors for a time horizon of one trading day measured as an average of the absolute value of the DOS (in dollars)

Moneyness (S/X)	Models	$
<.94	JR	0.81
	MP Lower	0.29
	MP Upper	0.30
	BS	0.08
		(30)*
0.94-0.98	JR	1.80
	MP Lower	0.27
	MP Upper	0.31
	BS	0.13
		(45)
0.98-1.02	JR	2.09
	MP Lower	0.38
	MP Upper	0.40
	BS	0.35
		(55)
1.02-1.06	JR	1.89
	MP Lower	0.48
	MP Upper	0.49
	BS	0.39
		(51)
>1.06	JR	1.70
	MP Lower	0.32
	MP Upper	0.32
	BS	0.29
		(200)
All Moneyness	JR	1.72
	MP Lower	0.34
	MP Upper	0.35
	BS	0.28
		(381)

* The number of observations for each category is within brackets

Tables VII and VIII present the pricing errors for the out-of-sample test for the time horizon of one trading day. Table VII presents the average of the absolute value of the DOS as a percentage of the mid-point of the bid-ask price of the options for each moneyness category. Table VIII contains the average of the absolute value of the DOS in dollar terms for each moneyness category. Although the pricing errors appear to be fairly constant in dollars terms across moneyness (table VIII), a difference is observed between the out-of-the-money and the in-the-money options in percentage terms. From table VII it is shown that the pricing ability of all the models improves when the moneyness (S/X) increases. As in Tables IV and V, the Black-Scholes (1973) model outperforms the two models that are based on a lattice approach. Strikingly, the model of Masson-Perrakis (2000) outperforms the Jackwerth-Rubinstein (1996) model. This suggests that for predictions over short horizons (one trading day), the Masson-Perrakis (2000) model performs better than the Jackwerth-Rubinstein (1996) model. However, the results in Tables IV and V show that the latter performs better than the former for predictions over longer horizons (30, 60, and 90 days).

To summarize, the Black-Scholes (1973) outperforms the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models for prediction over any time horizon. However, the Black-Scholes (1973) model overvalues out-of-the-money calls. The Masson-Perrakis (2000) model is theoretically consistent with the market incompleteness and involves linear programming. Moreover, it is less sensitive to option mispricings than the Jackwerth-Rubinstein (1996) model. Its pricing ability on the options on the TSE 60 index is weak for predictions over long time horizons, but relatively strong for a one day time horizon when compared to the Jackwerth-Rubinstein (1996) model. The Jackwerth-Rubinstein (1996) model searches for the smoothest risk neutral distributions. It is sensitive to option mispricings and involves non-linear programming. Moreover, only locally optimized solutions are feasible with optimization software. Its ability to price options is better than the Masson-Perrakis (2000) model for long horizons, but worse than the Black-Scholes (1973) model (except for ratios of moneyness higher than 1.06).

Although the two methods that are based on a lattice approach incorporate the information contained in the smile via the risk neutral probability distribution recovered from a cross-section of option prices, the empirical results of this paper show that the Black-Scholes (1973) model remains a more accurate model to price options on the TSE-60 index.

This result for the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) model may be partially due to the model specification. Jackwerth (2000) states:" matching the option prices by minimizing the squared errors puts more weight on in-the-money compared to out-of-money options" (Jackwerth, 2000, p. 438) when discussing the model of Jackwerth and Rubinstein (1996). This explanation is also applicable to the Masson-Perrakis (2000) model because it gives more weight to the in-the-money options compared to the out-of-the-money options.

It is important to note that the number of steps used in the computation of the binomial trees for the Jackwerth-Rubinstein (1996) and the Masson-Perrakis (2000) methods are unequal. Binomial trees of 199 and 499 steps are used for these methods respectively. Theoretically, the larger the number of steps, the better the definition of the shape of the recovered risk neutral probability distribution(s). More precise pricing performances should be obtained with a larger number of binomial tree steps. However, since better results are obtained for predictions with long time horizons for the method of Jackwerth-Rubinstein (1996), computed with only 199 steps, the number of steps does not seem to be an important factor in explaining the difference in the pricing performances.

VI. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

This paper investigates the volatility smile in a Canadian context. Based on the implied volatilities of a large number of options on the TSE-60 index having different moneyness (S/X) and time-to-maturity, I provide evidence for the existence of a Canadian smile. The implied volatilities appear to increase for an increase of moneyness (S/X). That is, the smile is upward sloping when the moneyness is expressed as S/X. Also I find that the implied volatility appears to increase as the number of days to maturity decreases for deep in-the-money and deep out-of-money options. The presence of a volatility smile is inconsistent with the assumption of constant volatility of the Black-Scholes (1973) model. Therefore, the risk neutral probability distribution of the underling asset may not be lognormal.

For this reason, the risk neutral probability distributions are empirically recovered from option prices by using the Jackwerth-Rubinstein (1996) and Masson-Perrakis (2000) models. Then, using the derived distributions, out-of-sample options are priced. While comparing the pricing performance of the two above models with the Black-Scholes (1973) I find that overall the latter is the best model to price options. The out-of-sample pricing performance also suggests that for predictions over short horizons (one trading day), the Masson-Perrakis (2000) model performs better than the Jackwerth-Rubinstein (1996) model. However, for longer horizon (30, 60, and 90 days), the latter outperforms the Masson-Perrakis (2000) model.

In this paper I also verify the absence of arbitrage opportunity. Only arbitrage opportunities that are small in magnitude are found. If transaction cost were considered, such opportunity would probably not be profitable.

Although it has been found that the Black-Scholes (1973) model outperforms the two other methods based on lattice approach, the inconsistency of the Black-Scholes (1973) model remains because a volatility smile is empirically observable. Thus, this area of research on the smile is still of interest. Here follows recommendations and ideas for further research on the smile and on methods used to recover the risk neutral probability distribution.

As explained in the methodology, Jackwerth and Rubinstein (1996) have introduced the constraints of the optimization problem in the objective function with a penalty parameter. The authors advocate that such transformation is done to avoid overfitting the data when dealing with the midpoint of the bid-ask quotes instead of the bid and ask quotes. In this paper, the approach of Masson-Perrakis (2000) has been implemented without penalty parameter. However, it would be interesting to investigate if a better pricing performance could be obtained by introducing the constraints of the optimization problem in the objective function.

Also Jackwerth and Rubinstein (1996) use the same penalty parameter for all the constraints incorporated in the objective function. It may be relevant to use a different value of α for each of the constraints. This way a large value α could be associated to the constraints that have to be strictly satisfied. Similarly a small value of α could be associated to the constraints that can be satisfied with more flexibility.

Another venue for future research is based on the work of Jackwerth (2000). Recall that this author recovers the risk aversion across wealth of a representative investor. Two distributions were required to derive this risk aversion function: the risk neutral probability distributions and the objective distribution. The objective distribution can be estimated from historical returns on the index. For the risk neutral probability distribution, Jackwerth (2000) uses a modified version of the Jackwerth and Rubinstein (1996) model. Further research may consider an alternative approach to estimate the risk neutral probability distribution based on the Masson-Perrakis (2000) method. A comparison between the risk aversion recovered from the S&P 500 index and from the TSE 60 index and their respective options may be of interest.

Finally, this paper provides evidence that few arbitrage opportunities exist in the Canadian index option market. As discussed previously, the number of arbitrage opportunities appears to be larger in the U.S. market. It would be relevant to investigate the reason for such a difference in the number of arbitrage opportunities using intraday data for the Canadian and U.S markets.

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Web sites:

Montreal Exchange: www.me.org

Lindo Systems Inc.: http://www.lindo.com

VIII. APPENDICES

Appendix A

Observations deleted for the out-of-sample test for time horizons of 30,60, and 90 days

	Spot date to recover the probability distribution	Maturity of the option to priced (days)	Bid of the option to price	Ask of the option to price	Moneyness of the option to price (S/X)	Spot	Option value from Jackwerth-Rubinstein (1996)	Option value from the lower bound of Masson-Perrakis (2000)	Option value from the upper bound of Masson-Perrakis (2000)	Option Value from Black and Sholes
1	9/15/00	91	12.6	13.6	0.905	533.92	7.86	12.22	12.09	14.05
2	9/15/00	91	9.95	10.95	0.890	533.92	4.99	12.11	12.08	11.72
3	9/15/00	91	8.1	9.1	0.875	533.92	2.87	11.99	12.07	9.72
4	9/15/00	91	6.5	7.5	0.861	533.92	1.45	11.87	12.06	8.01
5	9/15/00	91	5.2	6.2	0.847	533.92	0.60	11.75	12.04	6.57
6	9/15/00	91	3.9	4.9	0.834	533.92	0.17	11.63	12.03	5.36
7	9/15/00	91	3.05	4.05	0.821	533.92	0.03	11.52	12.02	4.35
8	9/15/00	91	2.2	3.2	0.809	533.92	0.01	11.40	12.01	3.51
9	9/15/00	91	1.7	2.7	0.797	533.92	0.00	11.28	11.99	2.82
10	9/15/00	91	1.15	2.15	0.785	533.92	0.00	11.16	11.98	2.26
11	9/15/00	91	0.9	1.9	0.774	533.92	0.00	11.04	11.97	1.80
12	9/15/00	91	0.6	1.6	0.763	533.92	0.00	10.93	11.96	1.42
13	9/15/00	91	0.45	1.45	0.752	533.92	0.00	10.81	11.94	1.12
14	9/15/00	91	0.25	1.25	0.742	533.92	0.00	10.69	11.93	0.88
15	8/15/00	30	5.15	6.15	0.940	648.54	10.48	12.99	12.90	6.94
16	8/15/00	30	3.3	4.3	0.926	648.54	10.39	12.94	12.85	5.03
17	8/15/00	30	2.15	3.15	0.913	648.54	10.30	12.88	12.80	3.57
18	8/15/00	30	1.1	2.1	0.901	648.54	10.21	12.83	12.75	2.49
19	7/14/00	59	6	7	0.901	648.83	12.56	13.31	14.49	8.68
20	3/21/00	30	3.55	4.55	0.926	536.87	9.24	9.59	10.42	4.91
21	3/21/00	30	2.6	3.6	0.910	536.87	9.14	9.52	10.37	3.36
22	3/21/00	30	1.55	2.55	0.895	536.87	9.03	9.45	10.32	2.25
23	3/21/00	30	0.95	1.95	0.880	536.87	8.92	9.38	10.27	1.48
24	9/20/00	30	1.25	2.25	0.876	613.17	0.20	1.61	1.25	2.11
25	9/20/00	30	0.75	1.75	0.864	613.17	0.01	1.61	1.24	1.49
26	9/20/00	30	0.4	1.4	0.852	613.17	0.00	1.60	1.24	1.04
27	2/21/00	59	2.9	3.9	0.874	524.42	4.92	8.93	8.39	7.01
28	2/21/00	59	2.05	3.05	0.860	524.42	4.86	8.91	8.37	5.60
29	6/21/00	30	28.4	29.4	1.027	657.07	32.95	41.71	48.88	29.36
30	6/21/00	30	22.15	23.15	1.011	657.07	26.24	35.26	42.48	23.20
31	6/21/00	30	16.8	17.8	0.996	657.07	20.34	30.10	36.95	17.91
32	6/21/00	30	12	13	0.981	657.07	15.66	25.05	32.31	13.50
33	6/21/00	30	8.55	9.55	0.966	657.07	11.34	20.71	27.90	9.93
34	3/20/00	58	5.55	6.55	0.902	541.23	1.30	4.23	4.22	5.63

Appendix B

Observations deleted for the out-of-sample test for a time horizon of one trading day

	Spot date to recover the probability Distribution	Maturity of the option to priced (days)	Bid of the option to price	Ask of the option to price	Moneyness of the option to price (S/X)	Spot	Option value from Jackwerth-Rubinstein (1996)	Option value from the lower bound of Masson-Perrakis (2000)	Option value from the upper bound of Masson-Perrakis (2000)	Option Value from Black and Sholes
1	12/21/99	29	1.4	1.9	0.945	491.27	2.16	3.92	3.93	1.77
2	10/17/00	29	2.5	3.5	0.890	596.41	1.29	2.46	2.45	2.35
3	7/18/00	29	6.95	7.95	0.958	651.51	10.41	6.59	7.69	7.05
4	6/21/00	29	7.65	8.65	0.958	603.48	2.39	6.84	11.29	8.06
5	2/17/00	29	3.45	4.45	0.943	546.80	1.97	6.66	6.66	4.06

Appendix C

SAS Program to price a series of options at the time from a risk neutral probability distribution and the possible terminal index prices

```
/* the risk neutral probability distributions p1, p2 p3 p4 p5 p6 and
the possible terminal index prices s1 s2 s3 s4 s5 s6 are inputs in this
program*/


data rub(keep=opt1 opt2 opt3 opt4 opt5 opt6 opt7 opt8 opt9 opt10 opt11
         opt12 opt13 opt14 opt15 opt16 opt17 opt18 opt19 opt20 opt21
         opt22 opt23 opt24 opt25);
        infile 'u:\ibouchard\input\prob.csv' dlm=','firstobs=1;
        input p1 s1 p2 s2 p3 s3 p4 s4 p5 s5 p6 s6;

/* the interest rate and the time-to-maturity and year are introduced
as follows*/

r=0.0545;
t=0.082191781;

/* the number written next to pr and ws have to be changed iteratively
to price a series of options from a different distribution and from
different terminal index prices*/

%let pr=1;
%let ws=1;

/*the 25 following lines introduce the strike price of the series of
options to price; the lines have to be modified for every series of
options*/

%let wx1    =     520   ;
%let wx2    =     530   ;
%let wx3    =     540   ;
%let wx4    =     550   ;
%let wx5    =     560   ;
%let wx6    =     570   ;
%let wx7    =     580   ;
%let wx8    =     590   ;
%let wx9    =     600   ;
%let wx10   =     610   ;
%let wx11   =     620   ;
%let wx12   =     630   ;
%let wx13   =     1000  ;
%let wx14   =     1000  ;
%let wx15   =     1000  ;
%let wx16   =     1000  ;
%let wx17   =     1000  ;
%let wx18   =     1000  ;
```

```
%let wx19    =     1000  ;
%let wx20    =     1000  ;
%let wx21    =     1000  ;
%let wx22    =     1000  ;
%let wx23    =     1000  ;
%let wx24    =     1000  ;
%let wx25    =     1000  ;


opt1=(p&pr*max(0, (s&ws-&wx1)))* (EXP(-(r*t)));
opt2=(p&pr*max(0, (s&ws-&wx2)))* (EXP(-(r*t)));
opt3=(p&pr*max(0, (s&ws-&wx3)))* (EXP(-(r*t)));
opt4=(p&pr*max(0, (s&ws-&wx4)))* (EXP(-(r*t)));
opt5=(p&pr*max(0, (s&ws-&wx5)))* (EXP(-(r*t)));
opt6=(p&pr*max(0, (s&ws-&wx6)))* (EXP(-(r*t)));
opt7=(p&pr*max(0, (s&ws-&wx7)))* (EXP(-(r*t)));
opt8=(p&pr*max(0, (s&ws-&wx8)))* (EXP(-(r*t)));
opt9=(p&pr*max(0, (s&ws-&wx9)))* (EXP(-(r*t)));
opt10=(p&pr*max(0, (s&ws-&wx10)))* (EXP(-(r*t)));
opt11=(p&pr*max(0, (s&ws-&wx11)))* (EXP(-(r*t)));
opt12=(p&pr*max(0, (s&ws-&wx12)))* (EXP(-(r*t)));
opt13=(p&pr*max(0, (s&ws-&wx13)))* (EXP(-(r*t)));
opt14=(p&pr*max(0, (s&ws-&wx14)))* (EXP(-(r*t)));
opt15=(p&pr*max(0, (s&ws-&wx15))) * (EXP(-(r*t)));
opt16=(p&pr*max(0, (s&ws-&wx16))) * (EXP(-(r*t)));
opt17=(p&pr*max(0, (s&ws-&wx17))) * (EXP(-(r*t)));
opt18=(p&pr*max(0, (s&ws-&wx18))) * (EXP(-(r*t)));
opt19=(p&pr*max(0, (s&ws-&wx19))) * (EXP(-(r*t)));
opt20=(p&pr*max(0, (s&ws-&wx20))) * (EXP(-(r*t)));
opt21=(p&pr*max(0, (s&ws-&wx21))) * (EXP(-(r*t)));
opt22=(p&pr*max(0, (s&ws-&wx22))) * (EXP(-(r*t)));
opt23=(p&pr*max(0, (s&ws-&wx23))) * (EXP(-(r*t)));
opt24=(p&pr*max(0, (s&ws-&wx24))) * (EXP(-(r*t)));
opt25=(p&pr*max(0, (s&ws-&wx25))) * (EXP(-(r*t)));

proc print data=rub;
run;

data rub2 (keep=opt1sum opt2sum opt3sum opt4sum opt5sum opt6sum opt7sum
           opt8sum opt9sum opt10sum opt11sum opt12sum opt13sum opt14sum
           opt15sum opt16sum opt17sum opt18sum opt19sum opt20sum
           opt21sum opt22sum opt23sum opt24sum opt25sum);
     set rub;
     opt1sum + opt1;
     opt2sum + opt2;
     opt3sum + opt3;
     opt4sum + opt4;
     opt5sum + opt5;
     opt6sum + opt6;
     opt7sum + opt7;
     opt8sum + opt8;
     opt9sum + opt9;
     opt10sum + opt10;
```

```
                  opt11sum + opt11;
                  opt12sum + opt12;
                  opt13sum + opt13;
                  opt14sum + opt14;
                  opt15sum + opt15;
                  opt16sum + opt16;
                  opt17sum + opt17;
                  opt18sum + opt18;
                  opt19sum + opt19;
                  opt20sum + opt20;
                  opt21sum + opt21;
                  opt22sum + opt22;
                  opt23sum + opt23;
                  opt24sum + opt24;
                  opt25sum + opt25;
run;

/* the last line of the output file, which is the cumulative sum of the
present value of the possible terminal payoff of each option of the
series, gives the option prices*/

proc print;

run;
```